Gevrey regularity for a system coupling the Navier-Stokes equations with a beam

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joint work with Takéo Takahashi

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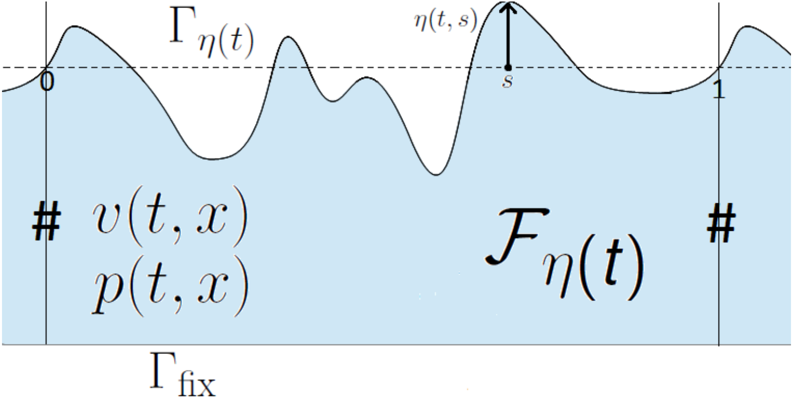
Plan

1. Presentation of the problem
2. Sketch of the proof
3. A control issue

Plan

2D fluid-structure interaction system



Fluid equations

- Navier-Stokes equations for the velocity of the fluid :

$$\begin{cases} \partial_t v + (v \cdot \nabla) v - \operatorname{div} \mathbb{T}(v,p) = 0 \quad \text{in } \mathcal{F}_{\eta(t)},\ t > 0 \\ \operatorname{div} v = 0 \quad \text{in } \mathcal{F}_{\eta(t)},\ t > 0 \\ v, p \text{ 1-periodic in the variable } x_1 \end{cases}$$

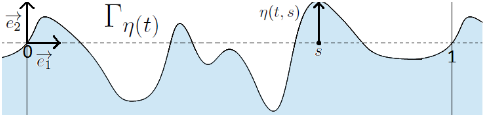
$$\text{where } \mathbb{T}(v,p) \overset{\text{def}}{=} 2\nu D(v) - p\,\mathrm{Id}, \quad D(v) \overset{\text{def}}{=} \frac{1}{2}\left((\nabla v) + {}^t(\nabla v)\right)$$

- Condition on the fixed part of the boundary :

$$v = 0 \quad \text{on } \Gamma_{\mathrm{fix}},\ t > 0.$$

Structure equation

- Structure interface : $\Gamma_{\eta(t)} = \{(s, \eta(t,s)) \mid s \in (0,1)\}$, $t > 0$.



- Interface condition :

$$v(t,s,\eta(t,s)) = (\partial_t \eta)(t,s)\overrightarrow{e_2},\ s \in (0,1), t > 0$$

- Vertical force exerted by the fluid on the structure :

$$\widetilde{F}_{\text{fluid}\to\text{str}}(v,p)(t,s) = -\sqrt{1 + |\partial_s \eta(t,s)|^2}\,[\mathbb{T}(v,p)n]\,(s,\eta(t,s)) \cdot \overrightarrow{e_2}$$

- Beam equation for the deformation ($\alpha > 0$ rigidity ; $\beta \geq 0$ stretching) :

$$\partial_{tt}\eta + \alpha\partial_{ssss}\eta - \beta\partial_{ss}\eta = \widetilde{F}_{\text{fluid}\to\text{str}}(v,p),\ s \in (0,1), t > 0,$$

η 1-periodic in the variable s.

Structure equation

- Compatibility condition :

$$0 = \int_{\mathcal{F}_{\eta(t)}} \operatorname{div}(v) dx = \int_{\partial \mathcal{F}_{\eta(t)}} v \cdot n d\gamma = \int_0^1 \partial_t \eta(t,s) ds = \frac{d}{dt} \int_0^1 \eta(t,s) ds$$

$$\rightsquigarrow \quad \int_0^1 \eta(t,s) ds = 0, t > 0.$$

- Then p is not determined up to a constant and satisfies

$$\int_0^1 \Big(p(t,s,\eta(t,s)) - 2\nu \Big\{ (1 + |\partial_s \eta|^2)^{1/2} \left[D(v) n \right] (t,s,1+\eta(t,s)) \cdot e_2 \Big\} \Big) ds = 0$$

- To hide this constraint we introduce :

$$M : L^2_\sharp(0,1) \to \left\{ f \in L^2_\sharp(0,1) \mid \int_0^1 f ds = 0 \right\}, \ F_{\text{fluid}\to\text{str}}(v,p) = M \widetilde{F}_{\text{fluid}\to\text{str}}(v,p)$$

$$\rightsquigarrow \quad \partial_{tt}\eta + \alpha \partial_{ssss}\eta - \beta \partial_{ss}\eta = F_{\text{fluid}\to\text{str}}(v,p), \ s \in (0,1), t > 0.$$

2D fluid-structure interaction system

$$\left\{\begin{array}{c}
\partial_t v + (v\cdot\nabla)v - \operatorname{div}\mathbb{T}(v,p) = 0 \quad \text{in } \mathcal{F}_{\eta(t)},\ t>0 \\
\operatorname{div} v = 0 \quad \text{in } \mathcal{F}_{\eta(t)},\ t>0 \\
v = \partial_t \eta \overrightarrow{e_2} \quad \text{on } \Gamma_{\eta(t)},\ t>0 \\
v = 0 \quad \text{on } \Gamma_{\text{fix}},\ t>0 \\
\partial_{tt}\eta + \alpha\partial_{ssss}\eta - \beta\partial_{ss}\eta = F_{\text{fluid}\to\text{str}}(v,p),\ s\in(0,1), t>0, \\
\eta \ \text{ 1-periodic in the variable s,} \quad \int_0^1 \eta\, ds = 0,\ t>0 \\
\eta(0) = \eta_1^0, \quad \partial_t\eta(0) = \eta_2^0, \quad v(0,x) = v^0(x) \quad x\in\mathcal{F}_{\eta_1^0}.
\end{array}\right.$$

with

$$F_{\text{fluid}\to\text{str}}(v,p)(t,s) = -M\sqrt{1+|\partial_s\eta(t,s)|^2}\,[\mathbb{T}(v,p)n]\,(s,\eta(t,s))\cdot\overrightarrow{e_2}$$

Definition of stong solutions and assumptions

Strong solution (v, η, p) on $[0, T]$:

$$v \in L^2(0, T; \mathbf{H}^2(\mathcal{F}_\eta) \cap C_b([0, T]; \mathbf{H}^1(\mathcal{F}_\eta)) \cap H^1(0, T; \mathbf{L}^2(\mathcal{F}_\eta)),$$

$$p \in L^2(0, T; H^1(\mathcal{F}_\eta))$$

$$\eta \in L^2(0, T; H^{7/2}_{\#}(0,1)) \cap C_b([0, T]; H^{5/2}_{\#}(0,1)) \cap H^1(0, T; H^{3/2}_{\#}(0,1)),$$

$$\partial_t \eta \in L^2(0, T; H^{3/2}_{\#}(0,1)) \cap C_b([0, T]; H^{1/2}_{\#}(0,1)) \cap H^1(0, T; (H^{1/2}_{\#}(0,1))'),$$

Assumptions on the initial conditions (AIC) :

$$\eta_1^0 \in H^{3+\varepsilon}_{\#}(0,1), \quad \eta_2^0 \in H^{1+\varepsilon}_{\#}(0,1), \quad \eta^0 > -1,$$

$$\int_0^1 \eta_1^0 ds = \int_0^1 \eta_2^0 ds = 0$$

$$v^0 \in \mathbf{H}^1(\mathcal{F}_{\eta_1^0}) \quad \text{and} \quad v^0 \ e_1 - \text{periodic},$$

$$\operatorname{div} v^0 = 0 \text{ in } \mathcal{F}_{\eta_1^0},$$

$$v^0(s, 1 + \eta_1^0(s)) = \eta_2^0(s)\overrightarrow{e_2} \quad s \in (0,1),$$

$$v^0 = 0 \quad \text{on } \Gamma_{\text{fix}}.$$

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$$v \in L^2(0,T;\mathbf{H}^2(\mathcal{F}_\eta)) \cap C_b([0,T];\mathbf{H}^1(\mathcal{F}_\eta)) \cap H^1(0,T;\mathbf{L}^2(\mathcal{F}_\eta)),$$

$$p \in L^2(0,T;H^1(\mathcal{F}_\eta))$$

$$\eta \in L^2(0,T;H^{7/2}_\#(0,1)) \cap C_b([0,T];H^{5/2}_\#(0,1)) \cap H^1(0,T;H^{3/2}_\#(0,1)),$$

$$\partial_t\eta \in L^2(0,T;H^{3/2}_\#(0,1)) \cap C_b([0,T];H^{1/2}_\#(0,1)) \cap H^1(0,T;(H^{1/2}_\#(0,1))'),$$

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$$v^0 = 0 \quad \text{on } \Gamma_{\text{fix}}.$$

Local existence theorem

Theorem

Assume that v^0, η_1^0, η_2^0 *satisfy (AIC). Then there exists* $C_0 > 0$ *such that if*

$$\left\|[v^0,\ \eta_1^0,\ \eta_2^0]\right\|_{\mathbf{H}^1(\mathcal{F}(\eta_1^0))\times H^{3+\varepsilon}_{\#}(0,1)\times H^{1+\varepsilon}_{\#}(0,1)} \leq C_0,$$

then there exists a strong solution (η, v, p) *on* $[0, +\infty)$ *such that*

$$\eta(t,\cdot) > -1 \quad t \in [0,\infty).$$

Theorem

Assume that v^0, η_1^0, η_2^0 *satisfy (AIC). Then there exists* $T > 0$ *and* $C_0 > 0$ *such that if*

$$\|\eta_1^0\|_{H^2(0,1)} \leq C_0$$

then there exists a strong solution (η, v, p) *on* $[0, T]$ *such that*

$$\eta(t,\cdot) > -1 \quad t \in [0,T].$$

$\rightsquigarrow$ B, Takahashi (SIMA 2019).

Local existence theorem

Theorem (B, Takahashi (2019))

Assume that v^0, η_1^0, η_2^0 satisfy (AIC). Assume also that

$$\eta_1^0 \in W^{7,\infty}(0,1).$$

Then there exists $T > 0$ and a strong solution (η, v, p) on $[0, T]$ such that

$$\eta(t,\cdot) > -1 \quad t \in [0,T].$$

$\rightsquigarrow$ B, Takahashi (just finalized !).

Open questions

- Initial deformation η_1^0 less regular ($\eta_1^0 \notin W^{7,\infty}(0,1)$) ?
- Initial regularity is not transported :
  - $\leadsto$ Blow up or contact if $T_{\max} < +\infty$?
  - $\leadsto$ Uniqueness ?
- Global existence ?
- Control issues ?

Context

The damped case

$$\partial_{tt}\eta \underbrace{-\delta\partial_{sst}\eta}_{damping} + \alpha\partial_{ssss}\eta - \beta\partial_{ss}\eta = F_{\text{fluid}\to\text{str}}(v,p). \quad (\delta > 0)$$

Model.

- Quarteroni, Tuveri, Veneziani (2000) (cardivascular model).

Well-posedness.

- Beirao da Veiga (2004), Chambolle-Desjardin-Esteban-Grandmont (2005), Lequeurre (2011, 2013), Grandmont- Hillairet (2016), Grandmont-Hillairet-Lequeurre (2019).

Stabilization.

- Raymond (2010), B-Takahashi (2017), Casanova (2018).

Context

The non damped case :

$$\partial_{tt}\eta + \alpha\partial_{ssss}\eta - \beta\partial_{ss}\eta = F_{\text{fluid}\to\text{str}}(v, p).$$

Well-posedness.

- Grandmont (2008) (Weak solutions),
- Grandmont- Hillairet-Lequeurre (2019) (Wave equation : $\alpha = 0$, $\beta > 0$).

Damped case v.s. non damped case

- Damped beam equation : $\alpha > 0$, $\beta \geq 0$ and $\delta > 0$,

$$\partial_{tt}\eta \underbrace{-\delta\partial_{sst}\eta}_{\textit{damping}} + \alpha\partial_{ssss}\eta - \beta\partial_{ss}\eta = F_{\text{fluid}\to\text{str}}(v, p).$$

Parabolic / Parabolic coupling $\Longrightarrow$ Parabolic

- Non damped beam equation $\alpha > 0$, $\beta \geq 0$,

$$\partial_{tt}\eta + \alpha\partial_{ssss}\eta - \beta\partial_{ss}\eta = F_{\text{fluid}\to\text{str}}(v, p).$$

Parabolic / Hyperbolic coupling $\Longrightarrow$????

Damped case v.s. non damped case

- Damped beam equation : $\alpha > 0$, $\beta \geq 0$ and $\delta > 0$,

$$\partial_{tt}\eta \underbrace{-\delta\partial_{sst}\eta}_{damping} + \alpha\partial_{ssss}\eta - \beta\partial_{ss}\eta = F_{\text{fluid}\to\text{str}}(v,p).$$

Parabolic / Parabolic coupling $\Longrightarrow$ Parabolic

- Non damped beam equation $\alpha > 0$, $\beta \geq 0$,

$$\partial_{tt}\eta \underbrace{-F_{\text{fluid}\to\text{str}}(v,p)}_{\text{damping}} + \alpha\partial_{ssss}\eta - \beta\partial_{ss}\eta = 0.$$

Parabolic / Hyperbolic coupling $\Longrightarrow$ Quasi-parabolic (Gevrey semigroup)

Outline

2D fluid-structure interaction system

$$\left\{\begin{array}{c}
\partial_t v + (v\cdot\nabla)v - \operatorname{div}\mathbb{T}(v,p) = 0 \quad \text{in } \mathcal{F}_{\eta(t)},\ t>0 \\
\operatorname{div} v = 0 \quad \text{in } \mathcal{F}_{\eta(t)},\ t>0 \\
v = (\partial_t\eta)\overrightarrow{e_2} \quad \text{on } \Gamma_{\eta(t)},\ t>0 \\
v = 0 \quad \text{on } \Gamma_{\text{fix}},\ t>0 \\
\partial_{tt}\eta - \delta\partial_{sst}\eta + \alpha\partial_{ssss}\eta - \beta\partial_{ss}\eta = F_{\text{fluid}\to\text{str}}(v,p),\ s\in(0,1), t>0, \\
\eta \text{ 1-periodic in the variable s}, \quad \int_0^1 \eta\, ds = 0,\ t>0 \\
\eta(0) = \eta_1^0, \quad \partial_t\eta(0) = \eta_2^0, \quad v(0,x) = v^0(x) \quad x\in\mathcal{F}_{\eta_1^0}.
\end{array}\right.$$

$$\alpha > 0,\ \beta \geq 0$$

$\delta > 0$ (damped case) or $\delta = 0$ (non damped case).

Structure equation

- State space and operators :

$$\mathcal{H}_S \overset{\text{def}}{=} \left\{ f \in L^2_\sharp(0,1) \mid \int_0^1 f ds = 0 \right\}$$

$$A_1\eta = \alpha\partial_{ssss}\eta - \beta\partial_{ss}\eta \quad \mathcal{D}(A_1) = H^4_\sharp(0,1) \cap \mathcal{H}_S,$$

$$\begin{cases} A_2\eta = -\delta\partial_{ss}\eta, \quad \mathcal{D}(A_2) = H^2_\sharp(0,1) \cap \mathcal{H}_S = \mathcal{D}(A_1^{1/2}) & \text{if } \delta > 0, \\ A_2\eta = 0 & \text{if } \delta = 0, \end{cases}$$

- Structure equation :

$$\partial_{tt}\eta + A_2\partial_t\eta + A_1\eta = F_{\text{fluid}\to\text{str}}(v,p),\ t > 0.$$

Structure equation

- Boundary operator :

$$\Lambda_\eta : \mathcal{H}_S \to \mathbf{L}^2(\partial\mathcal{F}_\eta), \quad (\Lambda_\eta \xi)(t, y_1, y_2) = \begin{cases} \xi(t, y_1)\overrightarrow{e_2} & \text{if } (y_1, y_2) \in \Gamma_\eta, \\ 0 & \text{if } (y_1, y_2) \in \Gamma_{\text{fix}}. \end{cases}$$

- Boundary conditions and force exerted by the fluid :

$$v = 0 \ \text{ on } \ \Gamma_{\text{fix}} \quad \text{and} \quad v(t, s, \eta(t, s)) = (\partial_t \eta)(t, s)\overrightarrow{e_2}, \ s \in (0, 1), t > 0$$

$$\rightsquigarrow v = \Lambda_\eta \partial_t \eta \ \text{ on } \partial\mathcal{F}_{\eta(t)}, t > 0$$

$$F_{\text{fluid}\to\text{str}}(v, p)(t, s) = -M\sqrt{1 + |\partial_s \eta(t, s)|^2}\,[\mathbb{T}(v, p)n]\,(s, \eta(t, s)) \cdot \overrightarrow{e_2}$$

$$\rightsquigarrow F_{\text{fluid}\to\text{str}}(v, p) = -\Lambda_\eta^* \mathbb{T}(v, p)n$$

- Structure equation :

$$\partial_{tt}\eta + A_2 \partial_t \eta + A_1 \eta = -\Lambda_\eta^* \mathbb{T}(v, p)n, \ t > 0.$$

2D fluid-structure interaction system

$$\left\{\begin{array}{c} \partial_t v + (v\cdot\nabla)v - \operatorname{div}\mathbb{T}(v,p) = 0 \quad \text{in } \mathcal{F}_{\eta(t)},\ t>0 \\ \operatorname{div} v = 0 \quad \text{in } \mathcal{F}_{\eta(t)},\ t>0 \\ v = \Lambda_\eta \partial_t \eta \quad \text{on } \partial\mathcal{F}_{\eta(t)},\ t>0 \\ \partial_{tt}\eta + A_2\partial_t\eta + A_1\eta = -\Lambda_\eta^* \mathbb{T}(v,p)n,\ t>0, \\ \eta(0) = \eta_1^0, \quad \partial_t\eta(0) = \eta_2^0, \quad v(0,x) = v^0(x) \quad x\in\mathcal{F}_{\eta_1^0}. \end{array}\right.$$

Change of variables

$$X : \mathcal{F}_{\eta_1^0} \to \mathcal{F}_{\eta(t)}, \quad X(t, y_1, y_2) \stackrel{\text{def}}{=} \left(y_1, \frac{y_2}{1 + \zeta(t, y_1)} \right),$$

$$\text{with } \zeta(t, y_1) \stackrel{\text{def}}{=} \frac{\eta(t, y_1) - \eta_1^0(y_1)}{1 + \eta_1^0(y_1)}$$

New velocity and pressure :

$$w(t, y) \stackrel{\text{def}}{=} \mathrm{Cof}(\nabla X(t, y))^* v(t, X(t, y)) \quad \text{and} \quad q(t, y) \stackrel{\text{def}}{=} p(t, X(t, y)).$$

Remark : The above definition with $\eta_1^0 = 0$ corresponds to a flat reference geometry for the structure (case treated in B, Takahashi (SIMA 2019)).

System rewritten in a fixed domain

$$
\left\{
\begin{array}{c}
\partial_t w - \operatorname{div} \mathbb{T}(w, q) = f(\zeta, w, q) \quad \text{in } \mathcal{F}_{\eta_1^0}, \ t > 0 \\[1em]
\operatorname{div} w = 0 \quad \text{in } \mathcal{F}_{\eta_1^0}, \ t > 0 \\[1em]
v = \Lambda_{\eta_1^0} \partial_t \eta \quad \text{on } \partial\mathcal{F}_{\eta_1^0}, \ t > 0 \\[1em]
\partial_{tt}\eta + A_2 \partial_t \eta + A_1 \eta = -\Lambda^*_{\eta_1^0} \mathbb{T}(w, q) n + g(\zeta, w, q), \ t > 0, \\[1em]
\eta(0) = \eta_1^0, \quad \partial_t \eta(0) = \eta_2^0, \quad w(0, x) = w^0(x) \quad x \in \mathcal{F}_{\eta_1^0}.
\end{array}
\right.
$$

$$\rightsquigarrow \quad Y' = AY + F, \quad Y(0) = Y_0 \in H$$

System rewritten in a fixed domain

$$
\begin{cases}
\partial_t w - \operatorname{div} \mathbb{T}(w,q) = f \quad \text{in } \mathcal{F}_{\eta_1^0},\ t > 0 \\
\operatorname{div} w = 0 \quad \text{in } \mathcal{F}_{\eta_1^0},\ t > 0 \\
v = \Lambda_{\eta_1^0} \partial_t \eta \quad \text{on } \mathcal{F}_{\eta_1^0},\ t > 0 \\
\partial_{tt}\eta + A_2 \partial_t \eta + A_1 \eta = -\Lambda^*_{\eta_1^0} \mathbb{T}(w,q)n + g,\ t > 0, \\
\eta(0) = \eta_1^0, \quad \partial_t \eta(0) = \eta_2^0, \quad w(0,x) = w^0(x) \quad x \in \mathcal{F}_{\eta_1^0}.
\end{cases}
$$

$$\rightsquigarrow \quad Y' = AY + F, \quad Y(0) = Y_0 \in H$$

System rewritten in a fixed domain

With $\eta_1 = \eta$ and $\eta_2 = \partial_t \eta$ the system rewrites :

$$\begin{cases} \partial_t w - \operatorname{div} \mathbb{T}(w,q) = f \quad \text{in } \mathcal{F}_{\eta_1^0},\ t > 0 \\ \operatorname{div} w = 0 \quad \text{in } \mathcal{F}_{\eta_1^0},\ t > 0 \\ v = \Lambda_{\eta_1^0} \eta_2 \quad \text{on } \partial\mathcal{F}_{\eta_1^0},\ t > 0 \\ \partial_t \eta_1 = \eta_2,\ t > 0, \\ \partial_t \eta_2 + A_2 \eta_2 + A_1 \eta_1 = -\Lambda^*_{\eta_1^0} \mathbb{T}(w,q) n + g,\ t > 0, \\ \eta_1(0) = \eta_1^0, \quad \eta_2(0) = \eta_2^0, \quad w(0,x) = w^0(x) \quad x \in \mathcal{F}_{\eta_1^0}. \end{cases}$$

$$\rightsquigarrow \quad Y' = AY + F, \quad Y(0) = Y_0 \in H$$

System rewritten in a fixed domain

$$
\begin{cases}
\partial_t w - \operatorname{div} \mathbb{T}(w,q) = f \quad \text{in } \mathcal{F}_{\eta_1^0},\ t>0 \\
\operatorname{div} w = 0 \quad \text{in } \mathcal{F}_{\eta_1^0},\ t>0 \\
v = \Lambda_{\eta_1^0}\eta_2 \quad \text{on } \partial\mathcal{F}_{\eta_1^0},\ t>0 \\
\partial_t \eta_1 = \eta_2,\ t>0, \\
\partial_t \eta_2 + A_2\eta_2 + A_1\eta_1 = -\Lambda^*_{\eta_1^0}\mathbb{T}(w,q)n + g,\ t>0, \\
\eta_1(0) = \eta_1^0, \quad \eta_2(0) = \eta_2^0, \quad w(0,x) = w^0(x) \quad x \in \mathcal{F}_{\eta_1^0}.
\end{cases}
$$

$$\rightsquigarrow \quad Y' = AY + F, \quad Y(0) = Y_0 \in H$$

Functional framework

- State space

$$H \stackrel{\text{def}}{=} \left\{ \begin{bmatrix} w \\ \eta_1 \\ \eta_2 \end{bmatrix} \in \mathbf{L}^2(\mathcal{F}_{\eta_1^0}) \times \mathcal{D}(A_1^{1/2}) \times \mathcal{H}_S \;\middle|\; \begin{array}{l} w \cdot n = \Lambda_{\eta_1^0}\eta_2 \cdot n \ \text{ on } \partial\mathcal{F}_{\eta_1^0}, \\ \operatorname{div} w = 0 \ \text{ in } \mathcal{F}_{\eta_1^0}, \end{array} \right\}$$

- Orthogonal projection operator

$$P_H : \mathbf{L}^2(\mathcal{F}_{\eta_1^0}) \times \mathcal{D}(A_1^{1/2}) \times \mathcal{H}_S \to H$$

- Linear operator

$$\mathcal{D}(A) \stackrel{\text{def}}{=} \left\{ \begin{bmatrix} w \\ \eta_1 \\ \eta_2 \end{bmatrix} \in \mathbf{H}^2(\mathcal{F}_{\eta_1^0}) \times \mathcal{D}(A_1) \times \mathcal{D}(A_1^{1/2}) \;\middle|\; \begin{array}{l} w = \Lambda_{\eta_1^0}\eta_2 \ \text{ on } \partial\mathcal{F}_{\eta_1^0}, \\ \operatorname{div} w = 0 \ \text{ in } \mathcal{F}_{\eta_1^0}, \end{array} \right\}$$

$$A \begin{bmatrix} w \\ \eta_1 \\ \eta_2 \end{bmatrix} \stackrel{\text{def}}{=} P_H \begin{bmatrix} \nu \Delta w \\ \eta_2 \\ -A_2\eta_2 - A_1\eta_1 - \Lambda_{\eta_1^0}^* [D(w)n] \end{bmatrix}$$

A generates a stable strongly continuous semigroup $(e^{tA})_{t>0}$ on H

Linearized system

Moreover, with

$$Y = \begin{bmatrix} w \\ \eta_1 \\ \eta_2 \end{bmatrix}, \quad Y_0 = \begin{bmatrix} w_0 \\ \eta_1^0 \\ \eta_2^0 \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} f \\ 0 \\ g \end{bmatrix}$$

the system rewrites

$$Y' = AY + F, \quad Y(0) = Y_0.$$

But to use a fixed point argument we need sufficiently strong regularity estimates for the above nonhomogeneous equation...

The damped case :

$A_2 = -\delta\partial_{ss}$, $\delta > 0$

The damped case $A_2 = -\delta\partial_{ss}$, $\delta > 0$

Key point :

$$t \in (0, +\infty) \mapsto e^{tA} \in H \text{ is analytic,}$$

or equivalently,

there exist $C > 0$, $\alpha > 0$ such that :

$$\forall \lambda \in \mathbb{C}^+, |\lambda| > \alpha, \quad \|(\lambda - A)^{-1}\|_{\mathcal{L}(H)} \leq \frac{C}{|\lambda|}.$$

It implies maximal regularity result :

$$\|Y'\|_{L^2(0,T;H)} + \|AY\|_{L^2(0,T;H)} + \|(-A)^{1/2} Y\|_{C([0,T];H)} \leq C(\|F\|_{L^2(0,T;H)} + \|(-A)^{1/2} Y_0\|_H),$$

which is sufficient to conclude with a fixed point argument...

The damped case $A_2 = -\delta\partial_{ss}$, $\delta > 0$

$$(\lambda - A)[v, \eta_1, \eta_2] = [f, g, h]$$

$$\Updownarrow$$

$$\begin{cases} \lambda v - \operatorname{div} \mathbb{T}(v, p) = f \quad \text{in } \mathcal{F}_{\eta_1^0}, \\ \operatorname{div} v = 0 \quad \text{in } \mathcal{F}_{\eta_1^0}, \\ v = \Lambda_{\eta_1^0} \eta_2 \quad \text{on } \partial\mathcal{F}_{\eta_1^0}, \\ \lambda \eta_1 - \eta_2 = g \\ \lambda \eta_2 + A_2 \eta_2 + \Lambda^*_{\eta_1^0} \mathbb{T}(v, p) n + A_1 \eta_1 = h. \end{cases}$$

The resolvent estimate to prove is

$$|\lambda| \|[v, \eta_1, \eta_2]\|_H \leq C \|[f, g, h]\|_H.$$

The damped case $A_2 = -\delta\partial_{ss}$, $\delta > 0$

With

$$f = 0 \quad g = 0$$

and

$$L(\lambda)\eta_1 \overset{\text{def}}{=} \Lambda^*_{\eta_1^0}\mathbb{T}(v_{\eta_1}, p_{\eta_1})n \;\text{ where }\; \begin{cases} \lambda v_{\eta_1} - \operatorname{div}\mathbb{T}(v_{\eta_1}, p_{\eta_1}) = 0 & \text{in } \mathcal{F}_{\eta_1^0}, \\ \operatorname{div} v_{\eta_1} = 0 & \text{in } \mathcal{F}_{\eta_1^0}, \\ v_{\eta_1} = \Lambda_{\eta_1^0}\eta_1 & \text{on } \partial\mathcal{F}_{\eta_1^0}. \end{cases}$$

The resolvent equation reduces to

$$\lambda^2\eta_1 + \lambda A_2\eta_1 + \lambda L(\lambda)\eta_1 + A_1\eta_1 = h.$$

Moreover, the resolvent estimate to prove reduces to

$$|\lambda|^2\|\eta_1\|_{\mathcal{H}_S} \leq C\|h\|_{\mathcal{H}_S}.$$

The damped case $A_2 = -\delta\partial_{ss}$, $\delta > 0$

Write

$$\lambda^2\eta_1 + \lambda A_2\eta_1 + A_1\eta_1 = h - \lambda L(\lambda)\eta_1,$$

and use a perturbation argument thanks to :

[Chen Triggiani, 1989]

If A_2 satisfies $\rho_1 A_1^{1/2} \leq A_2 \leq \rho_2 A_1^{1/2}$, $(0 < \rho_1 < \rho_2 < +\infty)$, then there exists $C > 0$ such that for all $\lambda \in \mathbb{C}^+$ the solution of

$$\lambda^2\eta_1 + \lambda A_2\eta_1 + A_1\eta_1 = h,$$

satisfies

$$|\lambda|^2\|\eta_1\|_{\mathcal{H}_S} + \|A_1\eta_1\|_{\mathcal{H}_S} \leq C\|h\|_{\mathcal{H}_S}.$$

As a consequence, the semigroup associated to the the underlying beam equation is analytic.

Indeed, we have

$$\|\lambda L(\lambda)\eta_1\| \leq C|\lambda|^{-1/2+\epsilon}(|\lambda|^2\|\eta_1\|_{\mathcal{H}_S} + \|A_1\eta_1\|_{\mathcal{H}_S})$$

and we conclude with $|\lambda|$ large enough.

The non damped case :

$$A_2 = 0$$

The non damped case $A_2 = 0$

Key point :

There exist $C > 0$, $\alpha > 0$ such that :

$$\forall \lambda \in \mathbb{C}^+, \lambda > \alpha, \quad \|(\lambda - A)^{-1}\|_{\mathcal{L}(H)} \leq \frac{C}{|\lambda|^{1/2}}$$

which implies

for all $\gamma > 2$, $t \in (0, +\infty) \mapsto e^{tA} \in H$ is of Gevrey class γ

($t \in (0, +\infty) \mapsto e^{tA} \in H$ *is of Gevrey class* γ means that for all compact $K \subset (0, +\infty)$ there exist $C, R > 0$, such that $\forall t \in K$, $\left\| \frac{d^n e^{tA}}{dt^n} \right\|_H \leq CR^n (n!)^\gamma$.)

The above estimate implies the following regularity result :

$$\|Y'\|_{L^2(0,T;H)} + \|AY\|_{L^2(0,T;H)} + \|(-A)^{1/2} Y\|_{C([0,T];H)} \leq C(\|(-A)^{1/2} F\|_{L^2(0,T;H)} + \|(-A)^{3/4+\varepsilon/2} Y_0\|_H).$$

... and we conclude with a fixed point argument...

The non damped case $A_2 = 0$

$$(\lambda - A)[v, \eta_1, \eta_2] = [f, g, h]$$

$$\Updownarrow$$

$$\begin{cases} \lambda v - \operatorname{div} \mathbb{T}(v, p) = f \quad \text{in } \mathcal{F}_{\eta_1^0}, \\ \operatorname{div} v = 0 \quad \text{in } \mathcal{F}_{\eta_1^0}, \\ v = \Lambda_{\eta_1^0} \eta_2 \quad \text{on } \partial\mathcal{F}_{\eta_1^0}, \\ \lambda \eta_1 - \eta_2 = g \\ \lambda \eta_2 + \Lambda_{\eta_1^0}^* \mathbb{T}(v, p) n + A_1 \eta_1 = h. \end{cases}$$

The resolvent estimate to prove is

$$|\lambda|^{1/2} \|[v, \eta_1, \eta_2]\|_H \leq C \|[f, g, h]\|_H.$$

The non damped case $A_2 = 0$

With

$$f = 0 \quad g = 0$$

and

$$L(\lambda)\eta_1 \stackrel{\text{def}}{=} \Lambda^*_{\eta_1^0}\mathbb{T}(v_{\eta_1}, p_{\eta_1})n \text{ where } \begin{cases} \lambda v_{\eta_1} - \operatorname{div}\mathbb{T}(v_{\eta_1}, p_{\eta_1}) = 0 & \text{in } \mathcal{F}_{\eta_1^0}, \\ \operatorname{div} v_{\eta_1} = 0 & \text{in } \mathcal{F}_{\eta_1^0}, \\ v_{\eta_1} = \Lambda_{\eta_1^0}\eta_1 & \text{on } \partial\mathcal{F}_{\eta_1^0}. \end{cases}$$

The resolvent equation reduces to

$$\lambda^2\eta_1 + \lambda L(\lambda)\eta_1 + A_1\eta_1 = h.$$

Moreover, the resolvent estimate to be proved reduces to

$$|\lambda|^{3/2}\|\eta_1\|_{\mathcal{H}_S} \leq C\|h\|_{\mathcal{H}_S}.$$

The non damped case $A_2 = 0$

From an integration by parts we first observe that

$$\langle L(\lambda)\eta_1, \eta_1\rangle_{\mathcal{H}_S} = \lambda \underbrace{\int_{\mathcal{F}_{\eta_1^0}} |v_{\eta_1}|^2 dy}_{\langle K(\lambda)\eta_1,\eta_1\rangle_{\mathcal{H}_S}} + \underbrace{\int_{\mathcal{F}_{\eta_1^0}} |\nabla v_{\eta_1}|^2 dy}_{\langle G(\lambda)\eta_1,\eta_1\rangle_{\mathcal{H}_S}} .$$

$$\rightsquigarrow L(\lambda) = \lambda K(\lambda) + G(\lambda)$$

with $K(\lambda)$ and $G(\lambda)$ self-adjoint satisfying

$$0 < K(\lambda) \leq \rho_2 A_1^{-1/4} \quad \text{and} \quad \rho_1 A_1^{1/4} \leq G(\lambda) \leq \rho_2 \left(A_1^{1/4} + |\lambda| A_1^{-1/4}\right)$$

$$\rightsquigarrow \quad \lambda^2(\eta_1 + K(\lambda)\eta_1) + \lambda G(\lambda)\eta_1 + A_1\eta_1 = h.$$

A (tricky) perturbation argument allows to replace $G(\lambda)$ by $A_1^{1/4}$ and consider

$$\lambda^2(\eta_1 +) + \lambda A_1^{1/4}\eta_1 + A_1\eta_1 = h.$$

The non damped case $A_2 = 0$

From an integration by parts we first observe that

$$\langle L(\lambda)\eta_1, \eta_1\rangle_{\mathcal{H}_S} = \lambda \underbrace{\int_{\mathcal{F}_{\eta_1^0}} |v_{\eta_1}|^2 dy}_{\langle K(\lambda)\eta_1, \eta_1\rangle_{\mathcal{H}_S}} + \underbrace{\int_{\mathcal{F}_{\eta_1^0}} |\nabla v_{\eta_1}|^2 dy}_{\langle G(\lambda)\eta_1, \eta_1\rangle_{\mathcal{H}_S}} .$$

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A (tricky) perturbation argument allows to replace $G(\lambda)$ by $A_1^{1/4}$ and consider

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The non damped case $A_2 = 0$

From an integration by parts we first observe that

$$\langle L(\lambda)\eta_1, \eta_1\rangle_{\mathcal{H}_S} = \lambda \underbrace{\int_{\mathcal{F}_{\eta_1^0}} |v_{\eta_1}|^2 dy}_{\langle K(\lambda)\eta_1,\eta_1\rangle_{\mathcal{H}_S}} + \underbrace{\int_{\mathcal{F}_{\eta_1^0}} |\nabla v_{\eta_1}|^2 dy}_{\langle G(\lambda)\eta_1,\eta_1\rangle_{\mathcal{H}_S}} .$$

$$\rightsquigarrow L(\lambda) = \lambda K(\lambda) + G(\lambda)$$

with $K(\lambda)$ and $G(\lambda)$ self-adjoint satisfying

$$0 < K(\lambda) \leq \rho_2 A_1^{-1/4} \quad \text{and} \quad \rho_1 A_1^{1/4} \leq G(\lambda) \leq \rho_2 \left(A_1^{1/4} + |\lambda| A_1^{-1/4}\right)$$

$$\rightsquigarrow \quad \lambda^2(\eta_1 + K(\lambda)\eta_1) + \lambda G(\lambda)\eta_1 + A_1\eta_1 = h.$$

A (tricky) perturbation argument allows to replace $G(\lambda)$ by $A_1^{1/4}$ and consider

$$\lambda^2(\eta_1 + K(\lambda)\eta_1) + \lambda A_1^{1/4}\eta_1 + A_1\eta_1 = h.$$

The non damped case $A_2 = 0$

To simplify let consider the beam type resolvent equation :

$$\lambda^2\eta_1 + \lambda A_1^{1/4}\eta_1 + A_1\eta_1 = h.$$

We have the following result from the literature :

[Chen Triggiani, 1990]

Let $\alpha \in (0, 1/2)$. There exists $C > 0$ such that for all $\lambda \in \mathbb{C}^+$ the solution of

$$\lambda^2\eta_1 + \lambda A_1^{\alpha}\eta_1 + A_1\eta_1 = h,$$

satisfies

$$|\lambda|^{1+2\alpha}\|\eta_1\|_{\mathcal{H}_S} + |\lambda|^{-2\alpha}\|A_1\eta_1\|_{\mathcal{H}_S} \leq C\|\lambda^2\eta_1 + \lambda A_1^{\alpha}\eta_1 + A_1\eta_1\|_{\mathcal{H}_S}.$$

As a consequence, the semigroup associated to the the underlying beam equation is of **Gevrey** class $\gamma > 1/2\alpha$.

With $\alpha = 1/4$ the semigroup is of **Gevrey** class $\gamma > 2$ and the estimate is

$$|\lambda|^{3/2}\|\eta_1\|_{\mathcal{H}_S} + |\lambda|^{-1/2}\|A_1\eta_1\|_{\mathcal{H}_S} \leq C\|\lambda^2\eta_1 + \lambda A_2\eta_1 + A_1\eta_1\|_{\mathcal{H}_S}.$$

The non damped case $A_2 = 0$

Proof of the simplified beam type resolvent estimate

$$|\lambda|^{3/2}\|\eta_1\|^2_{\mathcal{H}_S}+|\lambda|^{-1/2}\|A_1\eta_1\|^2_{\mathcal{H}_S} \leq C\|\lambda^2\eta_1+\lambda A_1^{1/4}\eta_1+A_1\eta_1\|_{\mathcal{H}_S}, (\lambda \in \mathbb{C}^+, |\lambda| > 1).$$

Step 1 :

$$\|\lambda^2\eta_1 + \lambda A_1^{1/4}\eta_1 + A_1\eta_1\|^2_{\mathcal{H}_S} \geq \|\lambda^2\eta_1 + A_1\eta_1\|^2_{\mathcal{H}_S} + |\lambda|^2\|A_1^{1/4}\eta_1\|^2_{\mathcal{H}_S}$$

Step 2 :

$$\|\lambda^2\eta_1 + \lambda A_1^{1/4}\eta_1 + A_1\eta_1\|^2_{\mathcal{H}_S} \geq |\lambda|^4\|\eta_1\|^2_{\mathcal{H}_S} + \|A_1\eta_1\|^2_{\mathcal{H}_S} + |\lambda|^2\|A_1^{1/4}\eta_1\|^2_{\mathcal{H}_S}$$

The non damped case $A_2 = 0$

Proof of the simplified beam type resolvent estimate

$$|\lambda|^{3/2}\|\eta_1\|^2_{\mathcal{H}_S} + |\lambda|^{-1/2}\|A_1\eta_1\|^2_{\mathcal{H}_S} \leq C\|\lambda^2\eta_1 + \lambda A_1^{1/4}\eta_1 + A_1\eta_1\|_{\mathcal{H}_S}, (\lambda \in \mathbb{C}^+, |\lambda| > 1).$$

Step 1 :

$$\|\lambda^2\eta_1 + \lambda A_1^{1/4}\eta_1 + A_1\eta_1\|^2_{\mathcal{H}_S} \geq \|\lambda^2\eta_1 + A_1\eta_1\|^2_{\mathcal{H}_S} + |\lambda|^2\|A_1^{1/4}\eta_1\|^2_{\mathcal{H}_S}$$

Step 2 :

$$\|\lambda^2\eta_1 + \lambda A_1^{1/4}\eta_1 + A_1\eta_1\|^2_{\mathcal{H}_S} \geq |\lambda|^4\|\eta_1\|^2_{\mathcal{H}_S} + \|A_1\eta_1\|^2_{\mathcal{H}_S} + |\lambda|^2\|A_1^{1/4}\eta_1\|^2_{\mathcal{H}_S}$$

The non damped case $A_2 = 0$

Proof of the simplified beam type resolvent estimate

$$|\lambda|^{3/2}\|\eta_1\|^2_{\mathcal{H}_S}+|\lambda|^{-1/2}\|A_1\eta_1\|^2_{\mathcal{H}_S} \leq C\|\lambda^2\eta_1+\lambda A_1^{1/4}\eta_1+A_1\eta_1\|_{\mathcal{H}_S}, (\lambda \in \mathbb{C}^+, |\lambda| > 1).$$

Step 1 :

$$\|\lambda^2\eta_1 + \lambda A_1^{1/4}\eta_1 + A_1\eta_1\|^2_{\mathcal{H}_S} \geq \|\lambda^2\eta_1 + A_1\eta_1\|^2_{\mathcal{H}_S} + |\lambda|^2\|A_1^{1/4}\eta_1\|^2_{\mathcal{H}_S}$$

Step 2 :

$$\|\lambda^2\eta_1 + \lambda A_1^{1/4}\eta_1 + A_1\eta_1\|^2_{\mathcal{H}_S} \geq |\lambda|^4\|\eta_1\|^2_{\mathcal{H}_S} + \|A_1\eta_1\|^2_{\mathcal{H}_S} + |\lambda|^2\|A_1^{1/4}\eta_1\|^2_{\mathcal{H}_S} + 2\mathrm{Re}\left(\lambda^2\eta_1, A_1\eta_1\right)_{\mathcal{H}_S}$$

The non damped case $A_2 = 0$

Proof of the simplified beam type resolvent estimate

$$|\lambda|^{3/2}\|\eta_1\|^2_{\mathcal{H}_S}+|\lambda|^{-1/2}\|A_1\eta_1\|^2_{\mathcal{H}_S} \leq C\|\lambda^2\eta_1+\lambda A_1^{1/4}\eta_1+A_1\eta_1\|_{\mathcal{H}_S}, (\lambda \in \mathbb{C}^+, |\lambda| > 1).$$

Step 1 :

$$\|\lambda^2\eta_1 + \lambda A_1^{1/4}\eta_1 + A_1\eta_1\|^2_{\mathcal{H}_S} \geq \|\lambda^2\eta_1 + A_1\eta_1\|^2_{\mathcal{H}_S} + |\lambda|^2\|A_1^{1/4}\eta_1\|^2_{\mathcal{H}_S}$$

Step 2 :

$$\|\lambda^2\eta_1 + \lambda A_1^{1/4}\eta_1 + A_1\eta_1\|^2_{\mathcal{H}_S} \geq |\lambda|^4\|\eta_1\|^2_{\mathcal{H}_S} + \|A_1\eta_1\|^2_{\mathcal{H}_S} + |\lambda|^2\|A_1^{1/4}\eta_1\|^2_{\mathcal{H}_S}$$
$$- 2(|\lambda|^{1/2}\|A_1^{3/4}\eta_1\|_{\mathcal{H}_S})(|\lambda|^{3/2}\|A_1^{1/4}\eta_1\|_{\mathcal{H}_S})$$

The non damped case $A_2 = 0$

Proof of the simplified beam type resolvent estimate

$$|\lambda|^{3/2}\|\eta_1\|^2_{\mathcal{H}_S}+|\lambda|^{-1/2}\|A_1\eta_1\|^2_{\mathcal{H}_S} \leq C\|\lambda^2\eta_1+\lambda A_1^{1/4}\eta_1+A_1\eta_1\|_{\mathcal{H}_S}, (\lambda \in \mathbb{C}^+, |\lambda| > 1).$$

Step 1 :

$$\|\lambda^2\eta_1 + \lambda A_1^{1/4}\eta_1 + A_1\eta_1\|^2_{\mathcal{H}_S} \geq \|\lambda^2\eta_1 + A_1\eta_1\|^2_{\mathcal{H}_S} + |\lambda|^2\|A_1^{1/4}\eta_1\|^2_{\mathcal{H}_S}$$

Step 2 :

$$\begin{aligned}\|\lambda^2\eta_1 + \lambda A_1^{1/4}\eta_1 + A_1\eta_1\|^2_{\mathcal{H}_S} \geq |\lambda|^4\|\eta_1\|^2_{\mathcal{H}_S} + \|A_1\eta_1\|^2_{\mathcal{H}_S} + |\lambda|^2\|A_1^{1/4}\eta_1\|^2_{\mathcal{H}_S} \\ - C(|\lambda|^4\|\eta_1\|^2_{\mathcal{H}_S} + \|A_1\eta_1\|_{\mathcal{H}_S})^{1/2}(|\lambda|^3\|A_1^{1/4}\eta_1\|^2_{\mathcal{H}_S})^{1/2}\end{aligned}$$

The non damped case $A_2 = 0$

Proof of the simplified beam type resolvent estimate

$$|\lambda|^{3/2}\|\eta_1\|^2_{\mathcal{H}_S}+|\lambda|^{-1/2}\|A_1\eta_1\|^2_{\mathcal{H}_S} \leq C\|\lambda^2\eta_1+\lambda A_1^{1/4}\eta_1+A_1\eta_1\|_{\mathcal{H}_S}, (\lambda \in \mathbb{C}^+, |\lambda| > 1).$$

Step 1 :

$$\|\lambda^2\eta_1 + \lambda A_1^{1/4}\eta_1 + A_1\eta_1\|^2_{\mathcal{H}_S} \geq \|\lambda^2\eta_1 + A_1\eta_1\|^2_{\mathcal{H}_S} + |\lambda|^2\|A_1^{1/4}\eta_1\|^2_{\mathcal{H}_S}$$

Step 2 :

$$\begin{aligned}
&\|\lambda^2\eta_1 + \lambda A_1^{1/4}\eta_1 + A_1\eta_1\|^2_{\mathcal{H}_S} \\
&\qquad \geq \frac{1}{2}\left(|\lambda|^4\|\eta_1\|^2_{\mathcal{H}_S} + \|A_1\eta_1\|^2_{\mathcal{H}_S}\right) + (|\lambda|^2 - c|\lambda|^3)\|A_1^{1/4}\eta_1\|^2_{\mathcal{H}_S}
\end{aligned}$$

The non damped case $A_2 = 0$

Proof of the simplified beam type resolvent estimate

$$|\lambda|^{3/2}\|\eta_1\|^2_{\mathcal{H}_S}+|\lambda|^{-1/2}\|A_1\eta_1\|^2_{\mathcal{H}_S} \leq C\|\lambda^2\eta_1+\lambda A_1^{1/4}\eta_1+A_1\eta_1\|_{\mathcal{H}_S}, (\lambda \in \mathbb{C}^+, |\lambda| > 1).$$

Step 1 :

$$\|\lambda^2\eta_1 + \lambda A_1^{1/4}\eta_1 + A_1\eta_1\|^2_{\mathcal{H}_S} \geq \|\lambda^2\eta_1 + A_1\eta_1\|^2_{\mathcal{H}_S} + |\lambda|^2\|A_1^{1/4}\eta_1\|^2_{\mathcal{H}_S}$$

Step 2 :

$$\|\lambda^2\eta_1+\lambda A_1^{1/4}\eta_1+A_1\eta_1\|^2_{\mathcal{H}_S}+c|\lambda|^3\|A_1^{1/4}\eta_1\|^2_{\mathcal{H}_S} \geq \frac{1}{2}\left(|\lambda|^4\|\eta_1\|^2_{\mathcal{H}_S} + \|A_1\eta_1\|^2_{\mathcal{H}_S}\right)$$

The non damped case $A_2 = 0$

Proof of the simplified beam type resolvent estimate

$$|\lambda|^{3/2}\|\eta_1\|^2_{\mathcal{H}_S}+|\lambda|^{-1/2}\|A_1\eta_1\|^2_{\mathcal{H}_S} \leq C\|\lambda^2\eta_1+\lambda A_1^{1/4}\eta_1+A_1\eta_1\|_{\mathcal{H}_S}, (\lambda \in \mathbb{C}^+, |\lambda| > 1).$$

Step 1 :

$$\|\lambda^2\eta_1 + \lambda A_1^{1/4}\eta_1 + A_1\eta_1\|^2_{\mathcal{H}_S} \geq \|\lambda^2\eta_1 + A_1\eta_1\|^2_{\mathcal{H}_S} + |\lambda|^2\|A_1^{1/4}\eta_1\|^2_{\mathcal{H}_S}$$

Step 2 :

$$\|\lambda^2\eta_1+\lambda A_1^{1/4}\eta_1+A_1\eta_1\|^2_{\mathcal{H}_S}+c|\lambda|^3\|A_1^{1/4}\eta_1\|^2_{\mathcal{H}_S} \geq \frac{1}{2}\left(|\lambda|^4\|\eta_1\|^2_{\mathcal{H}_S} + \|A_1\eta_1\|^2_{\mathcal{H}_S}\right)$$

Step 3 :

$$(1 + c|\lambda|)\|\lambda^2\eta_1 + \lambda A_1^{1/4}\eta_1 + A_1\eta_1\|^2_{\mathcal{H}_S} \geq \frac{1}{2}\left(|\lambda|^4\|\eta_1\|^2_{\mathcal{H}_S} + \|A_1\eta_1\|^2_{\mathcal{H}_S}\right)$$

The non damped case $A_2 = 0$

Proof of the simplified beam type resolvent estimate

$$|\lambda|^{3/2}\|\eta_1\|^2_{\mathcal{H}_S}+|\lambda|^{-1/2}\|A_1\eta_1\|^2_{\mathcal{H}_S} \leq C\|\lambda^2\eta_1+\lambda A_1^{1/4}\eta_1+A_1\eta_1\|_{\mathcal{H}_S}, (\lambda \in \mathbb{C}^+, |\lambda| > 1).$$

Step 1 :

$$\|\lambda^2\eta_1 + \lambda A_1^{1/4}\eta_1 + A_1\eta_1\|^2_{\mathcal{H}_S} \geq \|\lambda^2\eta_1 + A_1\eta_1\|^2_{\mathcal{H}_S} + |\lambda|^2\|A_1^{1/4}\eta_1\|^2_{\mathcal{H}_S}$$

Step 2 :

$$\|\lambda^2\eta_1+\lambda A_1^{1/4}\eta_1+A_1\eta_1\|^2_{\mathcal{H}_S}+c|\lambda|^3\|A_1^{1/4}\eta_1\|^2_{\mathcal{H}_S} \geq \frac{1}{2}\left(|\lambda|^4\|\eta_1\|^2_{\mathcal{H}_S} + \|A_1\eta_1\|^2_{\mathcal{H}_S}\right)$$

Step 3 :

$$C\|\lambda^2\eta_1 + \lambda A_1^{1/4}\eta_1 + A_1\eta_1\|^2_{\mathcal{H}_S} \geq \left(|\lambda|^3\|\eta_1\|^2_{\mathcal{H}_S} + |\lambda|^{-1}\|A_1\eta_1\|^2_{\mathcal{H}_S}\right)$$

The non damped case $A_2 = 0$

A precise study of $K(\lambda)$ permits to adapt the previous proof and obtain :

$$|\lambda|^{3/2}\|\eta_1\|_{\mathcal{H}_S} + |\lambda|^{-1/2}\|A_1\eta_1\|_{\mathcal{H}_S} \leq \|\lambda^2(\eta_1 + K(\lambda)\eta_1) + \lambda A_1^{1/4}\eta_1 + A_1\eta_1\|$$

$$K(\lambda)\eta = -\Lambda^*_{\eta_1^0}\mathbb{T}(\varphi_\eta, \pi_\eta)n \quad \left\{ \begin{array}{rl} \overline{\lambda}\varphi_\eta - \operatorname{div}\mathbb{T}(\varphi_\eta, \pi_\eta) = v_\eta & \text{in } \mathcal{F}_{\eta_1^0}, \\ \operatorname{div}\varphi_\eta = 0 & \text{in } \mathcal{F}_{\eta_1^0}, \\ \varphi_\eta = 0 & \text{on } \partial\mathcal{F}_{\eta_1^0}, \\ \lambda v_\eta - \operatorname{div}\mathbb{T}(v_\eta, p_\eta) = 0 & \text{in } \mathcal{F}_{\eta_1^0}, \\ \operatorname{div} v_\eta = 0 & \text{in } \mathcal{F}_{\eta_1^0}, \\ v_\eta = \Lambda_{\eta_1^0}\eta & \text{on } \partial\mathcal{F}_{\eta_1^0}, \end{array} \right.$$

provides, in particular, the following key estimates :

$$\|A_1^{\theta/2}K(\lambda)\eta\|_{\mathcal{H}_S} \leq C\left(\|A_1^{\theta/2-1/4}\eta\|_{\mathcal{H}_S} + |\lambda|^{\theta-1/2+\varepsilon}\|\eta\|_{\mathcal{H}_S}\right) \ (1/2 \leq \theta \leq 2).$$

$$\|[A_1^{3/8}, K(\lambda)]\eta\|_{\mathcal{H}_S} \leq C(|\lambda|^{-1}\|A_1^{1/2+\varepsilon}\eta\|_{\mathcal{H}_S} + |\lambda|^{\varepsilon}\|\eta\|_{\mathcal{H}_S})$$

The non damped case $A_2 = 0$

A precise study of $K(\lambda)$ permits to adapt the previous proof and obtain :

$$|\lambda|^{3/2}\|\eta_1\|_{\mathcal{H}_S} + |\lambda|^{-1/2}\|A_1\eta_1\|_{\mathcal{H}_S} \leq \|\lambda^2(\eta_1 + K(\lambda)\eta_1) + \lambda A_1^{1/4}\eta_1 + A_1\eta_1\|$$

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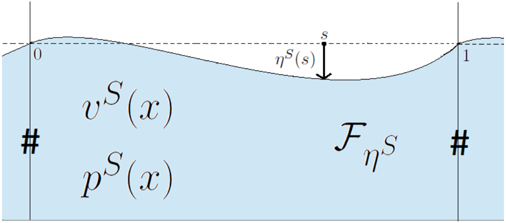
$$\|[A_1^{3/8}, K(\lambda)]\eta\|_{\mathcal{H}_S} \leq C(|\lambda|^{-1}\|A_1^{1/2+\varepsilon}\eta\|_{\mathcal{H}_S} + |\lambda|^{\varepsilon}\|\eta\|_{\mathcal{H}_S})$$

$$\rightsquigarrow \eta_1^0 \in W^{7,\infty}(0,1)$$

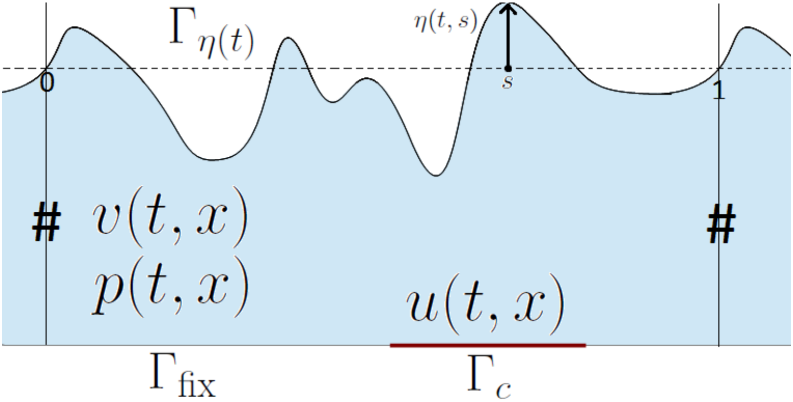
Outline

Target state



$$
\begin{cases}
(v^S \cdot \nabla) v^S - \operatorname{div} \mathbb{T}(v^S, p^S) = f^S \quad \text{in } \mathcal{F}_{\eta^S}, \\
\operatorname{div} v^S = 0 \quad \text{in } \mathcal{F}_{\eta^S} \\
v^S = 0 \quad \text{on } \Gamma_{\eta^S}, \\
v^S = b^S \quad \text{on } \Gamma_{\text{fix}}, \\
A_1 \eta^S = -\Lambda^*_{\eta^S} \mathbb{T}(v^S, p^S) n + M g^S.
\end{cases}
$$

Control problem



Find a control u such that $\lim_{t \to +\infty} (v(t), \eta(t)) = (v^S, \eta^S)$**,**

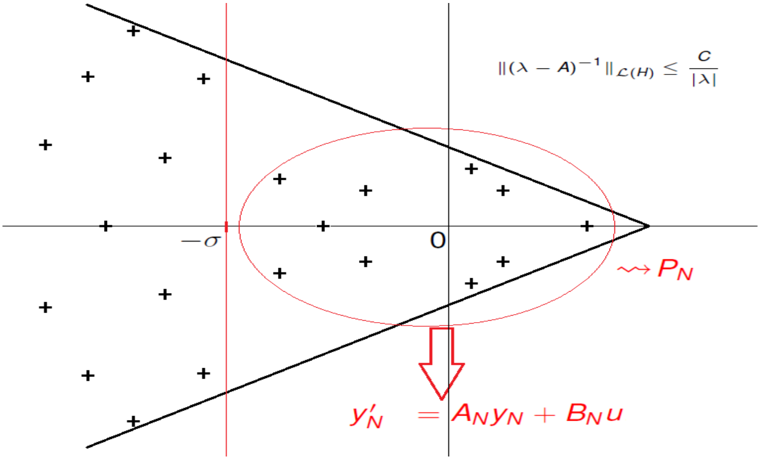
Control system

$$\left\{\begin{array}{l} \partial_t v + (v\cdot\nabla)v - \operatorname{div}\mathbb{T}(v,p) = f^S \quad \text{in } \mathcal{F}_{\eta(t)},\ t>0 \\ \operatorname{div} v = 0 \quad \text{in } \mathcal{F}_{\eta(t)},\ t>0 \\ v = (\partial_t\eta)\overrightarrow{e_2} \quad \text{on } \Gamma_{\eta(t)},\ t>0 \\ v = b^S + u \quad \text{on } \Gamma_{\text{fix}},\ t>0 \\ \partial_{tt}\eta + A_2\partial_t\eta + A_1\eta = -\Lambda_\eta^*\mathbb{T}(v,p)n + Mg^S,\ t>0, \\ \eta(0)=\eta_1^0, \quad \partial_t\eta(0)=\eta_2^0, \quad v(0,x)=v^0(x) \quad x\in\mathcal{F}_{\eta_1^0}. \end{array}\right.$$

Control system obtained by linearizing around $(v^S, 0, \eta^S)$:

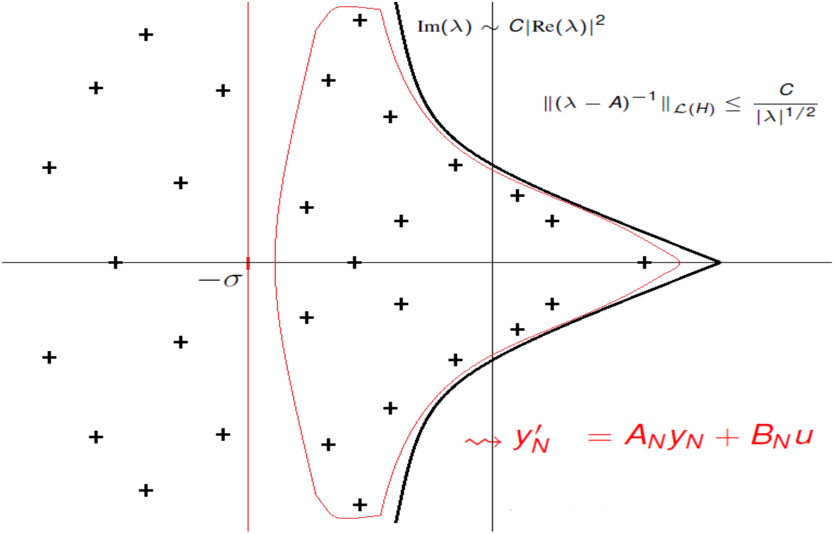
$$\rightsquigarrow \quad Y' = AY + Bu, \quad Y(0) = Y_0 \in H$$

Projected system (damped case $A_2 = -\delta\partial_{ss}$, $\delta > 0$)



$\rightsquigarrow$ B-Takahashi (2017) : Stabilization by finite dimensional feedback control.

Projected system (non damped case $A_2 = 0$)



$\rightsquigarrow$ work in progress ! ! !

Thank you for your attention !