Carleman-based reconstruction algorithm for the waves

Lucie Baudouin, LAAS, Toulouse
with M. de Buhan, S. Ervedoza & A. Osses.

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en l'honneur de Jean-Pierre Raymond.

Coefficient inverse problem in the wave equation

In a smooth bounded domain $\Omega \subset \mathbb{R}^n$, it writes for instance,

$$\begin{cases} \partial_{tt} y(t,x) - \Delta_x y(t,x) + p(x) y(t,x) = f(t,x), & (t,x) \in (0,T) \times \Omega, \\ y(t,x) = g(t,x), & (t,x) \in (0,T) \times \partial\Omega \\ (y(0,x), \partial_t y(0,x)) = (y^0(x), y^1(x)), & x \in \Omega. \end{cases}$$

or with variable speed

$$\begin{cases} \partial_{tt} y - \nabla \cdot (a(x) \nabla y) = f, & \text{in } (0,T) \times \Omega, \\ y = g, & \text{on } (0,T) \times \partial\Omega, \\ (y(0), \partial_t y(0)) = (y^0, 0), & \text{in } \Omega, \end{cases}$$

- Given data : Source terms f, g ; initial data : (y^0, y^1) ;
- Unknown : the potential $p = p(x)$ or the speed $a = a(x)$;
- Additional measurement : the flux $\partial_\nu y(t,x)$ on $(0,T) \times \partial\Omega$.

Motivation

- The determination in Ω of p or a from an additional measurement are inverse problems for which uniqueness and stability are well-known and proved using Carleman estimates.
- Classical reconstruction method : minimizing

$$J(p^k) = \|\partial_\nu y[p^k] - \partial_\nu y[p]\| \text{ or } J(a^k) = \|\partial_\nu y[a^k] - \partial_\nu y[a]\|$$

  generally not convex. $\rightsquigarrow$ May have several local minima.
  Algorithms not guaranteed to converge to the global minimum.
- Klibanov, Beilina and co-authors have worked a lot on related questions...

The Carleman-based reconstruction algorithm

• First goal : compute the PDE unknown coefficient with convergence estimates and no a priori first guess.

• Core idea : build a reconstruction algorithm

- using the structure of the proof of stability to prove the global convergence ;
- from the appropriate Carleman estimates to build the cost functional.

• Until now, the idea was applied to three reconstruction cases :

- potential / wave speed in the wave equation ;
- source term in a non linear heat equation by de Buhan, Schwindt & Boulakia.

Outline

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Determination of the potential in the wave equation

$$\begin{cases} \partial_{tt}y - \Delta y + py = f, & (0,T)\times\Omega, \\ y = g, & (0,T)\times\partial\Omega \\ (y(0),\partial_t y(0)) = (y^0,y^1), & \Omega. \end{cases}$$

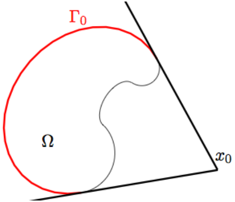
Is it possible to retrieve the potential $p = p(x)$, $x \in \Omega$ *from measurement of the flux* $\partial_\nu y(t,x)$ *on* $(0,T)\times\partial\Omega$ *?*

- Uniqueness : Given $p_1 \neq p_2$, can we guarantee $\partial_\nu y[p_1] \neq \partial_\nu y[p_2]$?
- Stability : If $\partial_\nu y[p_1] \simeq \partial_\nu y[p_2]$, can we guarantee that $p_1 \simeq p_2$?
- Reconstruction : Given $\partial_\nu y[p]$, can we compute p ?

• Known results : Uniqueness (Klibanov '92), stability (Yamamoto '99, Imanuvilov - Yamamoto '01), using Carleman estimates.

• Main question : Reconstruction ; how to compute the potential from the boundary measurement ?

Stability Result (Yamamoto '99, LB-Puel '01)



Let $x_0 \in \mathbb{R}^N \setminus \Omega$ and let Γ_0 and T satisfy

$$\{x \in \partial\Omega,\ (x - x_0)\cdot\nu(x) > 0\} \subset \Gamma_0 \quad ; \quad T > \sup_{x\in\Omega}\{|x - x_0|\}.$$

Let the potential p, the initial data y^0 and the solution $y[p]$ s.t.

$$\|p\|_{L^\infty(\Omega)} \le m, \quad \inf_{x\in\Omega}\{|y^0(x)|\} \ge \gamma > 0, \quad y[p] \in H^1(0,T;L^\infty(\Omega))$$

Then, one can prove uniqueness and local Lipschitz stability of the inverse problem for the wave equation : $\forall q \in L^\infty_{\le m}(\Omega)$,

$$\frac{1}{C}\,\|p - q\|_{L^2(\Omega)} \le \|\partial_\nu y[p] - \partial_\nu y[q]\|_{H^1((0,T);L^2(\Gamma_0))}\,.$$

Carleman estimate (LB, de Buhan, Ervedoza '13)

Assuming $q \in L^\infty_{\leq m}(\Omega)$, $L_q = \partial_{tt} - \Delta_x + q(x)$, $\varphi(t,x) = e^{\lambda(|x-x_0|^2 - \beta t^2)}$

$$\{x \in \partial\Omega,\, (x - x_0)\cdot\nu(x) > 0\} \subset \Gamma_0 \ , \ \sup_{x\in\Omega} |x - x_0| < \beta T$$

$\exists s_0 > 0$, $\lambda > 0$ and $M = M(s_0, \lambda, T, \beta, x_0, m) > 0$ such that

$$s \int_0^T\!\!\int_\Omega e^{2s\varphi}\left(|\partial_t w|^2 + |\nabla w|^2 + s^2|w|^2\right)\, dxdt \ + \ s^{1/2}\int_\Omega e^{2s\varphi(0)}|\partial_t w(0)|^2\, dx$$
$$\leq \ M \int_0^T\!\!\int_\Omega e^{2s\varphi}|L_q w|^2\, dxdt + Ms\int_0^T\!\!\int_{\Gamma_0} e^{2s\varphi}\,|\partial_\nu w|^2\ d\sigma dt,$$

for all $s > s_0$ and $w \in L^2(-T,T; H^1_0(\Omega))$ satisfying

$$\begin{cases} L_q w \in L^2(\Omega\times(-T,T)) \\ \partial_\nu w \in L^2((0,T)\times\Gamma_0), \\ w(0,x) = 0,\ \forall x \in \Omega. \end{cases}$$

$\rightsquigarrow$ but also Imanuvilov, Zhang, Klibanov,...

Towards a (re)constructive approach

It is easy to check that $Z = \partial_t (y[p] - y[q])$ satisfies

$$\begin{cases} \partial_{tt} Z - \Delta_x Z + q(x) Z = (q-p)\partial_t y[p], & (t,x) \in (0,T) \times \Omega, \\ Z(t,x) = 0, & (t,x) \in (0,T) \times \partial\Omega \\ (Z(0,x), \partial_t Z(0,x)) = (0, (q-p)y^0), & x \in \Omega. \end{cases}$$

Main idea : source term $(q-p)\partial_t y[p]$ less relevant than initial data $(q-p)y^0$, thanks to the Carleman estimate, whereas

$$\partial_\nu Z = \partial_t \partial_\nu y[p] - \partial_t \partial_\nu y[q] \quad \text{on } (0,T) \times \Gamma_0 \quad \text{is known.}$$

$\rightsquigarrow$ Hence, we try to fit Z using this information

Carleman based Reconstruction Algorithm

Initialization : $q^0 = 0$ or any initial guess.

Iteration : Given q^k,

1 - Compute $w[q^k]$ the solution of

$$\begin{cases} \partial_t^2 w - \Delta w + q^k w = f, & \text{in } \Omega \times (0,T), \\ w = g, & \text{on } \partial\Omega \times (0,T), \\ w(0) = w_0, \quad \partial_t w(0) = w_1, & \text{in } \Omega, \end{cases}$$

and set $\mu^k = \partial_t \left(\partial_\nu w[q^k] - \partial_\nu w[p]\right)$ on $\Gamma_0 \times (0,T)$.

2 - Introduce the functional

$$J_0^k(z) = \int_0^T \int_\Omega e^{2s\varphi} |L_{q^k} z|^2 + s \int_0^T \int_{\Gamma_0} e^{2s\varphi} |\partial_\nu z - \mu^k|^2,$$

on the space $\mathcal{T}^k = \{z \in L^2(0,T; H_0^1(\Omega)), z(t=0) = 0,$

$L_{q^k} z \in L^2(\Omega \times (0,T)), \partial_\nu z \in L^2(\Gamma_0 \times (0,T))\}.$

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Theorem

Assume the **geometric and time conditions***. Then, for all $s > 0$ and $k \in \mathbb{N}$, the functional J_0^k is continuous, strictly convex and coercive on $\mathcal{T}^k$ endowed with a suitable weighted norm.*

3 - Let Z^k be the unique minimizer of the functional J_0^k, and then set

$$\tilde{q}^{k+1} = q^k + \frac{\partial_t Z^k(0)}{w_0} \Leftrightarrow (\tilde{q}^{k+1} - q^k)w_0 = \partial_t Z^k(0),$$

where w_0 is the initial condition.

4 - Finally, set

$$q^{k+1} = T_m(\tilde{q}^{k+1}), \quad \text{where } T_m(q) = \begin{cases} q, & \text{if } |q| \leq m, \\ \text{sign}(q)m, & \text{if } |q| \geq m. \end{cases}$$

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Algorithm's convergence (LB, de Buhan & Ervedoza 13')

Theorem

Assuming the geometric and time conditions (among others), there exists a constant $M > 0$ *such that* $\forall s \geq s_0(m)$ *and* $k \in \mathbb{N}$,

$$\int_\Omega e^{2s\varphi(0)}(q^{k+1} - Q)^2 \, dx \leq \frac{M}{\sqrt{s}} \int_\Omega e^{2s\varphi(0)}(q^k - Q)^2 \, dx.$$

In particular, when s *is large enough, the algorithm converges.*

Remark : This algorithm converges to the global minimum from any initial guess.

Proof

The algorithm is based on the Bukhgeim-Klibanov method and uses $v^k = \partial_t \big(y[q^k] - y[p]\big)$ that solves

$$\begin{cases} \partial_t^2 v - \Delta v + q^k v = h^k, & \text{in } \Omega \times (0,T), \\ v = 0, & \text{on } \partial\Omega \times (0,T), \\ v(0) = 0, \quad \partial_t v(0) = (p - q^k) y^0, & \text{in } \Omega, \end{cases}$$

where $h^k = (p - q^k)\partial_t y[p]$.

By definition, $\mu^k = \partial_\nu v^k$ on $\Gamma_0 \times (0,T)$, and we notice that v^k is the unique minimizer of the functional :

$$J_h^k(w) = \int_0^T \int_\Omega e^{2s\varphi} |L_{q^k} w - h^k|^2 + s \int_0^T \int_{\Gamma_0} e^{2s\varphi} |\partial_\nu w - \mu^k|^2,$$

on the space $\mathcal{T}^k = \{w \in L^2(0,T; H_0^1(\Omega)), w(t=0) = 0,$
$L_{q^k} w \in L^2(\Omega \times (0,T)), \partial_\nu w \in L^2(\Gamma_0 \times (0,T))\}$.

Let us write the Euler Lagrange equations satisfied by :

Z^k minimizer of J_0^k

$$\int_0^T\int_\Omega e^{2s\varphi} L_{q^k} Z^k \, L_{q^k} w + s\int_0^T\int_{\Gamma_0} e^{2s\varphi}(\partial_\nu Z^k - \mu^k)\partial_\nu w = 0,$$

and v^k minimizer of J_h^k

$$\int_0^T\int_\Omega e^{2s\varphi}(L_{q^k} v^k - h^k) L_{q^k} w + s\int_0^T\int_{\Gamma_0} e^{2s\varphi}(\partial_\nu v^k - \mu^k)\partial_\nu w = 0,$$

for all $w \in \mathcal{T}^k$. Applying these to $w = Z^k - v^k$ and subtracting the two identities, we obtain :

$$\int_0^T\int_\Omega e^{2s\varphi}|L_{q^k} w|^2 + s\int_0^T\int_{\Gamma_0} e^{2s\varphi}|\partial_\nu w|^2 = \int_0^T\int_\Omega e^{2s\varphi} h^k \, L_{q^k} w,$$

implying ($2ab \le a^2 + b^2$)

$$\frac{1}{2}\int_0^T\int_\Omega e^{2s\varphi}|L_{q^k} w|^2 + s\int_0^T\int_{\Gamma_0} e^{2s\varphi}|\partial_\nu w|^2 \le \frac{1}{2}\int_0^T\int_\Omega e^{2s\varphi}|h^k|^2.$$

The *LHS* is precisely the *RHS* of the Carleman estimate. Hence :

$$s^{1/2}\int_\Omega e^{2s\varphi(0)}|\partial_t w(0)|^2\,dx \le M\int_0^T\int_\Omega e^{2s\varphi}|h^k|^2\,dxdt,$$

where $\partial_t w(0) = \partial_t Z^k(0) - \partial_t v^k(0)$. Moreover,

$$\partial_t Z^k(0) = (\tilde{q}^{k+1} - q^k)y^0, \quad \partial_t v^k(0) = (p - q^k)y^0, \quad h^k = (p - q^k)\partial_t y[p].$$

Therefore, since $\varphi(t) \le \varphi(0)$ for all $t \in (0, T)$ we have :

$$s^{1/2}\int_\Omega e^{2s\varphi(0)}|y^0|^2(\tilde{q}^{k+1}-p)^2\,dx \le M\|\partial_t y[p]\|^2_{L^2(0,T;L^\infty(\Omega))}\int_\Omega e^{2s\varphi(0)}(q^k-p)^2\,dx.$$

Using the positivity condition on y^0 and the fact that

$$|q^{k+1} - p| = |T_m(\tilde{q}^{k+1}) - T_m(p)| \le |\tilde{q}^{k+1} - p|$$

because T_m is Lipschitz and $T_m(p) = p$, we immediately deduce

$$\int_\Omega e^{2s\varphi(0)}(q^{k+1} - p)^2\,dx \le \left(\frac{M}{\sqrt{s}}\right)^{k+1}\int_\Omega e^{2s\varphi(0)}(q^0 - p)^2\,dx. \qquad \square$$

In theory, it works. But in practice ?

Two remarks :

- Discretizing the wave equation brings numerical artefacts...
- Minimizing a strictly convex and coercive quadratic functional based on a Carleman estimate means dealing with $e^{2se^{\lambda\psi}}$ for large parameters s and λ...

• New goal : propose a numerically efficient algorithm..

Ideas : We actually need an algorithm constructed with at least

- a regularization term in the cost functional,
- a single parameter Carleman estimate.

Natural idea for reconstruction

Given a continuous measurement $\mathscr{M}[p] = \partial_\nu y[p]|_{(0,T)\times\partial\Omega}$

- Discretize the wave equation

$$\begin{cases} \partial_{tt} y_h - \Delta_h y_h + p_h y_h = f_h \simeq f, \\ y_h|_{(0,T)\times\partial\Omega} = g_h \simeq g, \\ (y_h, \partial_t y_h)(t=0) = (y_h^0, y_h^1) \simeq (y^0, y^1). \end{cases}$$

- Solve the following discrete inverse problem : Find a potential p_h so that the corresponding discrete solution $y_h[p_h]$ approximates at best the measurement :

$$\partial_h y_h[p_h]|_{(0,T)\times\partial\Omega}(t,x) \simeq \mathscr{M}[p](t,x)$$

$$\text{i.e. } p_h = \text{Argmin}_{q_h} \|\partial_h y_h[q_h] - \mathscr{M}[p]\|_*$$

Question : Do we get $p_h \simeq p$?

Convergence of the discrete inverse problems

Remarks :

- Natural question for all inverse problems in infinite dimensions : Finding a source term, a conductivity...
- Depends *a priori* on the numerical scheme employed.

Main difficulty :

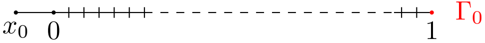
- Different dynamics for the continuous wave equation versus its discrete approximations, cf Ervedoza - Zuazua '11 :
  $\rightsquigarrow$ Numerical artefacts : High-frequency spurious waves.

Convergence results for the inverse problem :

- Penalization of high-frequencies with a regularization term in the discrete Carleman estimates.
- 1D (LB & Ervedoza '13) and 2D (LB & Ervedoza & Osses '15)

Numerical Simulations

- $\Omega = [0,1]$, $x_0 = -0.1$, $\Gamma_0 = \{x = 1\}$, $g = 0$, $\beta = 0.99$, $T = 1.5$, $\lambda = 0.1$, $s = 1$;



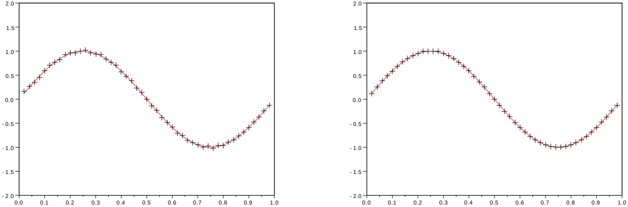
- Discretization with the finite-difference method : $N + 1 = \frac{1}{h}$, $(\Delta_h y_h)_j = \frac{y_{j+1} - 2y_j + y_{j-1}}{h^2}, \quad \forall j \in \{1, \cdots, N\}$
- Addition of a regularization term $s \int_0^T \int_0^1 e^{2s\varphi} |h \partial_h^+ \partial_t z_h|^2 \, dt$ to the cost functional J_0^k - from the discrete Carleman estimates - to have uniformity with respect to the discretization parameter h. Constraint : sh small enough.

  $\rightsquigarrow$ (LB & Ervedoza '13) and (LB & Ervedoza & Osses '15)

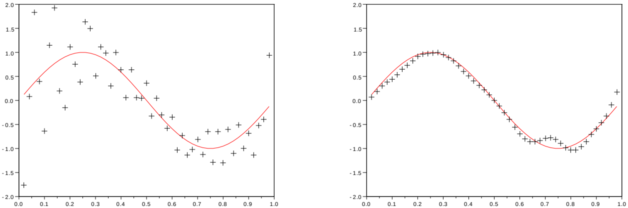
- Other approach : use high order finite elements to guarantee a conformal approximation (Cîndea-FernándezCara-Münch '13).

Without (left) and with (right) regularization term

- Without noise, for $p(x) = \sin(2\pi x)$, one has



- Noise parameter $\alpha = 10\%$ in the measurement : $(1 + \alpha\mathcal{N}(0,1))\mu$



New C-bRec algorithm

The algorithm is also modified according to the following items :

- Single parameter Carleman estimate ;
- Preconditioning of the cost functional ;
- Splitting of the observations by cut-off ;

... and the convergence result remains the same.

A single parameter Carleman estimate

(Lavrentiev Romanov Shishatskii '86)

Assuming the geometric condition on Γ_0, $L_q = \partial_{tt} - \Delta_x + q(x)$, $q \in L^\infty_{\leq m}(\Omega)$, $\sup_{x\in\Omega} |x - x_0| < \beta T$ and

$$\varphi(t,x) = |x - x_0|^2 - \beta t^2,$$

then $\exists s_0 > 0$ and $M = M(s_0, T, \beta, x_0, m) > 0$ such that

$$\underbrace{s^{1/2} \int_\Omega e^{2s\varphi(0)} |\partial_t w(0)|^2 \, dx}_{\text{initial energy}} \leq M \underbrace{\int_0^T\!\!\int_\Omega e^{2s\varphi} |L_q w|^2 \, dxdt}_{\text{source}}$$

$$+ \; Ms \underbrace{\int_0^T\!\!\int_{\Gamma_0} e^{2s\varphi} \, |\partial_\nu w|^2 \, d\sigma dt}_{\text{observation}} + Ms^3 \iint_{\{\varphi<0\}} e^{2s\varphi} |w|^2 \, dxdt$$

Preconditioning the new cost functional

Recalling the former

$$J_0^k(z) = \int_0^T \int_\Omega e^{2s\varphi}|L_{q^k} z|^2 + s\int_0^T \int_{\Gamma_0} e^{2s\varphi}|\partial_\nu z - \mu^k|^2,$$

we remove some exponential factors by introducing the conjugate variable $y = e^{\varphi} z$ in the new functional

$$\widetilde{J}_0^k(y) = \int_0^T \int_\Omega |\mathscr{L}_{s,q^k} y|^2 + s\int_0^T \int_{\Gamma_0} |\partial_\nu y - e^{2s\varphi}\mu^k|^2 + s^3 \iint_{\{\varphi<0\}} |y|^2,$$

which is minimized on the same set $\mathcal{T}^k$ as before and where the conjugate operator is $\mathscr{L}_{s,q} = e^{s\varphi}(\partial_t^2 - \Delta + q)e^{-s\varphi}$.

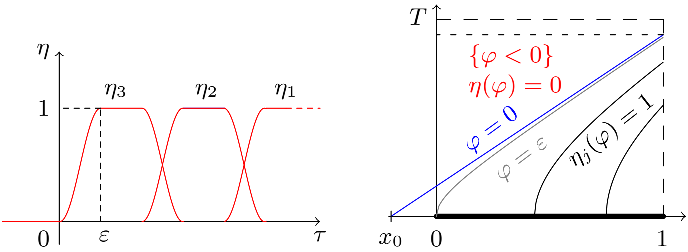
Nevertheless, there is still an exponential factor in the measurements.

Dealing finally with the observations

We split the observations in several slices and consider intervals in which the weight function does not significantly change. To do that :

$$\mu_j^k = \eta_j(\varphi)\mu^k, \qquad \forall \tau \in \mathbb{R}, \quad \sum_{j=1}^{N} \eta_j(\tau) = \eta(\tau),$$

where the η_j are the following cut-off functions ($\varepsilon = \inf_\Omega |x - x_0|^2$) :



Y_j minimizer of $\widetilde{J}_0^k[\mu_j^k] \quad \Rightarrow \quad Y = \sum_{j=1}^N Y_j$ minimizer of $\widetilde{J}^k[\mu^k]$.

Adapted C-bRec algorithm

Initialization : Any $q \in L^\infty_m(\Omega)$.

Iteration : Given q^k,

1 - Compute $y[q^k]$ the solution of $\begin{cases} \partial_t^2 y - \Delta y + q^k y = f, \\ y = g, \\ y(0) = y^0, \quad \partial_t y(0) = y^1, \end{cases}$

and for each j, set $\mu_j^k = \eta_j(\varphi)\partial_t \left(\partial_\nu y[q^k] - \mu\right)$ on $\Gamma_0 \times (0,T)$.

2 - Introduce the functional

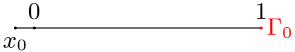
$$\widetilde{J}_0[\mu_j^k](y) = \int_0^T \int_\Omega |\mathscr{L} y|^2 + s \int_0^T \int_{\Gamma_0} |\partial_\nu y - \mu_j^k e^{s\varphi}|^2 + s^3 \iint_{\{\varphi<0\}} |y|^2.$$

3 - For each j, let Y_j be the unique minimizer of the functional $\tilde{J}_0[\mu_j^k]$, and then set $\widetilde{q}^{k+1} = q^k + \sum_j \frac{\partial_t Y_j(0)}{y^0 e^{s\varphi(0)}}$,

4 - Finally, set $q^{k+1} = T_m(\widetilde{q}^{k+1})$.

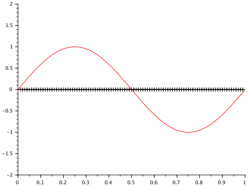
Discretization of the problem

- $\Omega = [0,1]$, $x_0 = -0.3$, $\Gamma_0 = \{x = 1\}$, $\beta = 0.99$, $T = 1.3$, $s = 100$, $f = 0$, $g = 2$, $u_0(x) = 2 + \sin(x\pi)$ and $u_1 = 0$.

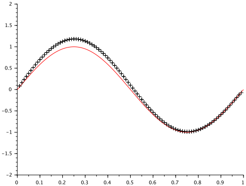


- To avoid the inverse crime, we use $\neq$ schemes and $\neq$ meshes in the direct and inverse problems :
  - direct problem : finite differences in space $h = 0.00025$, implicit theta scheme in time $\tau = 0.00033$;
  - inverse problem : finite differences in space $h = 0.05$, explicit Euler scheme in time $\tau = 0.05$, that is $CFL = 1$.

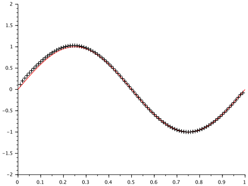
Illustration of the convergence of the algorithm



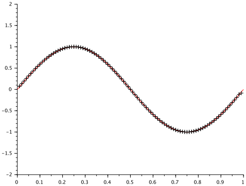
(a) q^0



(b) q^1

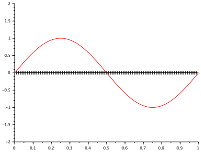


(c) q^2

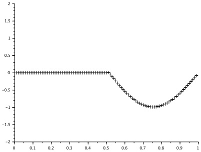


(d) q^3

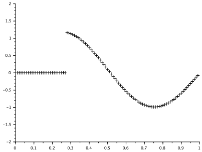
Illustration of the splitting



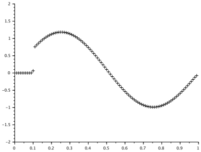
(e) $q_0^0 = q^0$



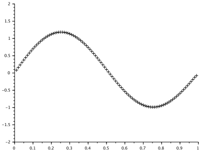
(f) q_1^0



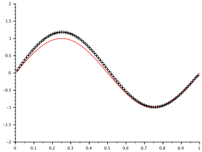
(g) q_2^0



(h) q_3^0

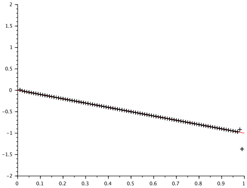


(i) q_4^0

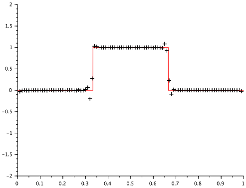


(j) $q_5^0 = q^1$

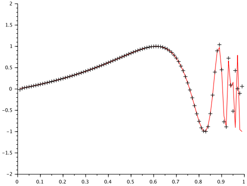
Other 1D simulations



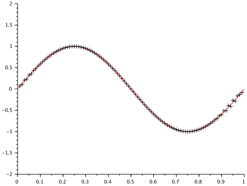
(a) $p = -x$



(b) $p = \text{gate}(x)$

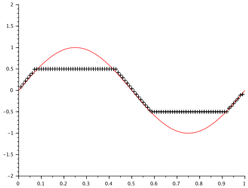


(c) $p(x) = \sin(\frac{x}{1-x})$

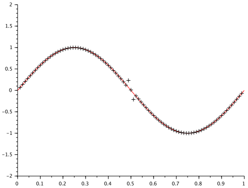


(d) $p(x) = \sin(2\pi x)$, with $q^0 = 10$

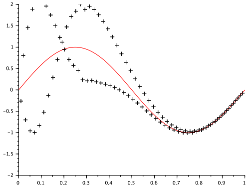
Wrong choices of the parameters



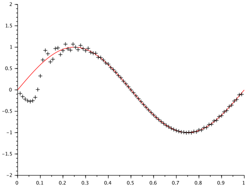
(a) Wrong choice of m



(b) y^0 vanishes at $x = 0.5$

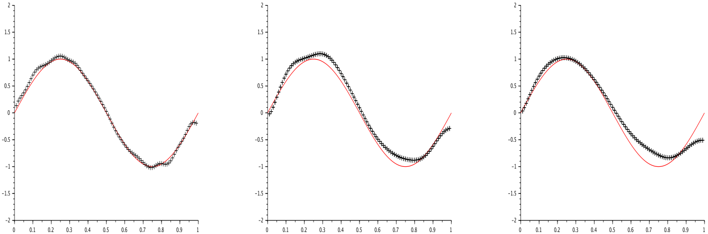


(c) No viscous term or sh too large



(d) $T = 0.9 < 1$

With noise on the measurement of the flux

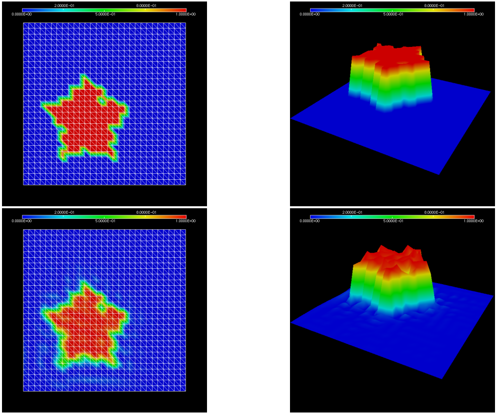


$s = 10$ and the noise is multiplicative : 1%, 5%, 10%.

Taking s too large seems to amplify the effects of the noise...

Numerical results in 2D

$\Omega = [0,1]^2$, $x_0 = (-0.3, -0.3)$ and $\Gamma_0 = \{x = 1\} \cup \{y = 1\}$



Exact potentials (top) vs Numerical potentials (bottom).

Outline

Recovery of the main coefficient

Wave equation with variable speed :

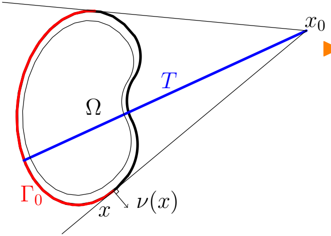
$$\begin{cases} \partial_{tt} y - \nabla \cdot (a(x) \nabla y) = f, & \text{in } (0,T) \times \Omega, \\ y = g, & \text{on } (0,T) \times \partial\Omega, \\ y(0) = y^0, \quad \partial_t y(0) = y^1, & \text{in } \Omega, \end{cases}$$

- Given data : Source terms (f, g), initial data : (y^0, y^1),
  boundary values $a = \mathbf{a}$ and $\partial_\nu a = \mathbf{a}_\nu$ on $\partial\Omega$.
- Unknown : the speed $a = a(x) > 0$, inside Ω.
- Additional measurement : the flux $\partial_\nu y(t, x)$ on $(0,T) \times \partial\Omega$.

Goal : Find the variable speed $a = a(x)$.

$\rightsquigarrow$ Application in medical imaging.

Setting and assumptions



- Geometric and time conditions :

  $\exists\, x_0 \notin \overline{\Omega}$, such that

  $$\Gamma_0 \supset \{x \in \partial\Omega,\ (x - x_0) \cdot \nu(x) \geq 0\},$$

  $$T > \frac{\sup_{x \in \Omega} |x - x_0|}{\sqrt{\alpha_0 \rho_0}}.$$

- Regularity assumption $y[a] \in H^2(0, T; W^{2,\infty}(\Omega))$.
- Initial conditions : $|\nabla y^0(x) \cdot (x - x_0)| \geq r_0 > 0$ and $y^1 = 0$ in Ω.
- $\mathcal{V}_{\mathbf{a},\mathbf{a}_\nu} = \{a \in C^1(\overline{\Omega}) \cap H^2(\Omega), \|\nabla a\|_{L^\infty(\Omega)} \leq m,\ 0 < \alpha_0 \leq a \leq \alpha_1,$
  $\nabla a \cdot (x - x_0) \leq 2(1 - \rho)a$ in $\Omega,\ a = \mathbf{a}, \partial_\nu a = \mathbf{a}_\nu$ on $\partial\Omega\}$.

Theorem (Inverse problem stability)

There exists a positive constant $M = M(\Omega, T, x_0, r_0, \mathbf{a}, \mathbf{a}_\nu, \alpha_0, \alpha_1)$ *such that for all* $a, \bar{a} \in \mathcal{V}_{\mathbf{a},\mathbf{a}_\nu}$ *:*

$$\|a - \bar{a}\|_{H_0^1(\Omega)} \leq M \|\partial_\nu y - \partial_\nu \bar{y}\|_{H^2(0,T;L^2(\Gamma_0))}.$$

Idea

The speed reconstruction algorithm is based on the fact that if $y[a]$, $y[a^k]$, are the solution of the wave equation, then

$$z^k = \partial_t^2 \left(y[a^k] - y[a] \right)$$

solves

$$\begin{cases} \partial_{tt} z^k - \nabla \cdot (a^k \nabla z^k) = g^k, & \text{in } (0,T) \times \Omega, \\ z^k = 0, & \text{on } (0,T) \times \partial\Omega, \\ z^k(0,\cdot) = z_0^k, \quad \partial_t z^k(0,\cdot) = 0, & \text{in } \Omega, \end{cases}$$

where

$$g^k = \nabla \cdot ((a^k - a)\nabla \partial_t^2 y[a]), \qquad z_0^k = \nabla \cdot ((a^k - a)\nabla w_0),$$

and for both operators (wave and first order) we can prove Carleman estimates.

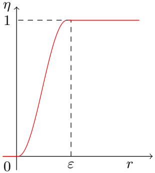
$\rightsquigarrow$ Holder stability results (Imanuvilov Yamamoto '03)

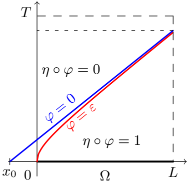
$\rightsquigarrow$ Lipschitz stability results (Klibanov Yamamoto '06)

$\rightsquigarrow$ Γ_0 small, Logarithmic stability (Bellassoued Yamamoto '06)

First step of the C-bRec algorithm

Minimization of





$$J_{s,a^k}[\mu](z) = \frac{1}{2}\int_0^T \int_\Omega e^{2s\varphi}|\partial_t^2 z - \nabla\cdot(a^k\nabla z)|^2$$

$$+\frac{s}{2}\int_0^T\int_{\Gamma_0} e^{2s\varphi}|\partial_\nu z - \mu|^2 + \frac{s}{2}\iint_{\{\varphi<0\}} e^{2s\varphi}\left(|\partial_t z|^2 + |\nabla z|^2 + s^2|z|^2\right)$$

$$+\frac{s}{2}\int_\Omega e^{2s\varphi(\pm T)}\left(\partial_t z(\pm T)^2 + |\nabla z(\pm T)|^2 + s^2 z(\pm T)^2\right)$$

in order to approximate $\tilde{z}^k = \eta(\varphi)z^k$, that satisfies :

- $\tilde{z}^k(0,\cdot) = \eta(\varphi(0,\cdot))z_0^k = \nabla\cdot((a^k - a)\nabla y^0)$;
- $\tilde{z}^k = \eta(\varphi)z^k = 0$ in $\{\varphi < 0\}$; $\tilde{z}^k(\pm T,\cdot) = 0$ because T large ;
- $\partial_\nu \tilde{z}^k = \tilde{\mu}^k$ in $(0,T)\times\Gamma_0$.

Second step

Then, we need to study the first order differential equation that encapsulate $a^k - a$.
One possibility is to solve the system

$$\begin{cases} \nabla \cdot (\delta a(x) \nabla y_0(x)) = -\tilde{z}^k(0,x), & \text{for } x \in \Omega, \\ \delta a = 0, & \text{on } \Gamma_{\nabla y_0} \subset \partial\Omega. \end{cases}$$

Another possibility is to work from the minimization of

$$K_{s,k}(\delta a) = \frac{1}{2} \int_\Omega e^{2s\varphi(0,\cdot)} |\nabla(\nabla \cdot (\delta a \nabla y_0)) - \nabla \tilde{z}^k(0,\cdot)|^2 dx$$

on $\left\{\delta a \in H_0^1(\Omega), \nabla \delta a \cdot \nabla w_0 \in H_0^1(\Omega)\right\}$, in order to approximate a.

First Tool : Carleman estimate for the waves

(Klibanov-Timonov '04, LB-deBuhan-Ervedoza-Osses '19)

Under the previous assumptions on x_0, Γ_0, T, $y[a]$, (y^0, y^1) and using a less restrictive admissible set $\mathcal{V}$,

$\exists \rho_0 > 0, \forall \beta \in (0, \alpha_0 \rho_0), \exists s_0 > 0, \exists C > 0, \forall s \geq s_0, \forall a \in \mathcal{V},$

$$\int_\Omega e^{2s\varphi(0)} \left(\partial_t v(0)^2 + |\nabla v(0)|^2 + s^2 v(0)^2 \right) dx$$

$$\leq C \int_{-T}^{T} \int_\Omega e^{2s\varphi} (\partial_t^2 v - \nabla \cdot (a \nabla v))^2 \, dx dt + Cs \int_{-T}^{T} \int_{\Gamma_0} e^{2s\varphi} |\partial_\nu v|^2 \, d\sigma dt$$

$$+ Cs \iint_{\{\varphi < 0\}} e^{2s\varphi} \left((\partial_t v)^2 + |\nabla v|^2 + s^2 v^2 \right) dx dt$$

$$+ Cs \int_\Omega e^{2s\varphi(\pm T)} \left(\partial_t v(\pm T)^2 + |\nabla v(\pm T)|^2 + s^2 v(\pm T)^2 \right) dx,$$

for all $v \in L^2((-T,T); H_0^1(\Omega))$, $\partial_\nu v \in L^2((-T,T) \times \partial\Omega)$, $\partial_t^2 v - \nabla \cdot (a \nabla v) \in L^2((-T,T) \times \Omega)$, where φ denotes the weight function $\varphi(t,x) = |x - x_0|^2 - \beta t^2$.

Second Tool : Carleman estimate for transport

(Klibanov-Yamamoto '06)

Let $x_0 \notin \overline{\Omega}$ and X be a vector field such that

$$X \in W^{2,\infty}(\Omega;\mathbb{R}^d) \cap C^0(\overline{\Omega};\mathbb{R}^d), \text{ and } \inf_{x\in\Omega} \{|X(x)\cdot(x-x_0)|\} > 0,$$

and set $\gamma_X = \text{sign}(X(x)\cdot(x-x_0))$, $\Gamma_X = \left\{x \in \partial\Omega, (X\cdot\nu)\gamma_X > 0\right\}$.

Then $\exists s_0 > 0, \exists C > 0$ such that $\forall s \geq s_0$,

$$\int_\Omega e^{2s|x-x_0|^2} \left(|\nabla(\nabla\cdot(bX))|^2 + s^2|\nabla b|^2 + s^4 b^2\right) dx$$
$$\leq C \int_\Omega e^{2s|x-x_0|^2} \left(|\nabla\left(\nabla\cdot(bX)\right)|^2 + s^2|\nabla\cdot(bX)|^2\right) dx$$

for any $b \in H^1_X(\Omega)$ satisfying $\nabla\cdot(bX) \in H^1_X(\Omega)$ where
$H^1_X(\Omega) = \left\{b \in H^1(\Omega), b = 0 \textit{ on } \Gamma_X\right\}$.

$\rightsquigarrow$ To be applied to $X = \nabla y_0$.

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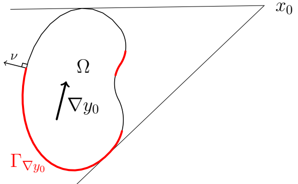
$\rightsquigarrow$ To be applied to $X = \nabla y_0$.



Algorithm

We have access to the measurement $\mu = \partial_\nu y[a]$ for a belonging to the admissible set

$$\mathcal{V}^*_{\mathbf{a},\mathbf{b}_\nu} := \big\{a \in W^{1,\infty}(\Omega),\ \nabla\cdot(a\nabla w_0) \in H^1(\Omega),\ \|\nabla a\|_{L^\infty(\Omega)} \leq m,$$
$$0 < \alpha_0 \leq a \leq \alpha_1 \text{ and } \nabla a\cdot(x - x_0) \leq 2(1-\rho)a \text{ in } \Omega,$$
$$a = \mathbf{a} \text{ and } \nabla a\cdot\nabla w_0 = \mathbf{b}_\nu \text{ on } \partial\Omega\big\},$$

Initialization : Any $a^0 \in \mathcal{V}^*_{\mathbf{a},\mathbf{b}_\nu}$.
Iteration : Given a^k,

1 - Compute $y[a^k]$ the solution of

$$\begin{cases} \partial_t^2 y - \nabla\cdot(a^k\nabla y) = f, & \text{in } \Omega\times(0,T), \\ y = g, & \text{on } \partial\Omega\times(0,T), \\ y(0) = y^0, \quad \partial_t y(0) = 0, & \text{in } \Omega, \end{cases}$$

and set $\mu^k = \eta(\varphi)\partial_t^2\left(\partial_\nu y[a^k] - \mu\right)$ on $\Gamma_0\times(0,T)$.

2 - Introduce the functional

$$\widetilde{J}_0[\mu^k](y) = \frac{1}{2}\int_0^T\int_\Omega |\mathcal{L}y|^2\,dxdt + \frac{s}{2}\int_0^T\int_{\Gamma_0} |\partial_\nu y - \mu^k e^{s\varphi}|^2\,d\sigma dt$$
$$+ \frac{s}{2}\iint_{\{\varphi<0\}} \left(|\partial_t y|^2 + |\nabla y|^2 + s^2|y|^2\right)\,dxdt$$
$$+ \frac{s}{2}\int_\Omega \left(|\partial_t y|^2 + |\nabla y|^2 + s^2|y|^2\right)(\pm T)dx + \text{ regularization term}$$

on the trajectories $y \in L^2(0,T;H_0^1(\Omega))$, $\partial_\nu y \in L^2((0,T)\times\Gamma_0)$, $\partial_t^2 y - \nabla\cdot(a^k\nabla y) \in L^2((0,T)\times\Omega)$ and $\partial_t y(0,\cdot) = 0$ in Ω, and where $\mathcal{L}y = e^{s\varphi}(\partial_t^2 - \nabla\cdot(a^k\nabla))(e^{-s\varphi}y)$ is the conjugate operator.

Lemma

Assume the **geometric and time conditions***. Then, for all $s > 0$, the functional $\widetilde{J}_0$ is continuous, strictly convex and coercive on $\mathcal{T}$ endowed with a suitable weighted norm.*

3 - Let Y^k be the unique minimizer of the functional $\widetilde{J}_0[\mu^k]$ on $\mathcal{T}$. In particular, $Y^k(0,\cdot) \in H_0^1(\Omega)$. Then minimize the functional

$$K_{s,k}(\delta a) = \frac{1}{2}\int_\Omega |\nabla(\nabla \cdot (\delta a \nabla w_0)) - \nabla Y^k(0,\cdot)|^2 dx$$

on $\left\{\delta a \in H_0^1(\Omega), \nabla \delta a \cdot \nabla w_0 \in H_0^1(\Omega)\right\}$, and denote its unique minimizer by δa^k.

Then we set $\tilde{a}^{k+1} = a^k + \delta a^k$.

4 - Finally, set

$$a^{k+1} = T_{\mathbf{a},\mathbf{b}_\nu}\left(\tilde{a}^{k+1}\right),$$

where $T_{\mathbf{a},\mathbf{b}_\nu}$ is the projection on the admissible set $\mathcal{V}^*_{\mathbf{a},\mathbf{b}_\nu}$.

Convergence result

The set $\mathcal{V}^*_{\mathbf{a},\mathbf{b}_\nu}$ is closed and convex for the topology induced by the norm $\|b\|_s^2 = \int_\Omega e^{2s\varphi(0)} \left(s^2|\nabla b|^2 + s^4 b^2 + |\nabla\left(\nabla \cdot (b\nabla w_0)\right)|^2\right) dx.$

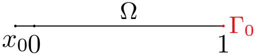
Theorem (LB-deBuhan-Ervedoza-Osses '19)

Assume the geometric and time conditions, the regularity assumption and the initial condition. Let $a \in \mathcal{V}^*_{\mathbf{a},\mathbf{b}_\nu}$.
There exists a constant $M > 0$ such that for all s large enough and for all $k \in \mathbb{N}$,

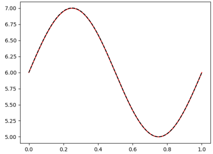
$$\|a^{k+1} - a\|_s^2 \leq \frac{C}{\inf\{s^2, e^{2s\inf_\Omega(\varphi(0)-\varepsilon)}\}} \|a^k - a\|_s^2.$$

In particular, for s large enough, $(a^k)_{k\in\mathbb{N}}$ strongly converges to a in the norm $\|\cdot\|_s$.

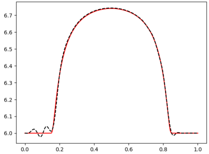
1D Numerical results

- $f = 0$, $T = 5$, $s = 100$, $y_0(x) = x - 1$, 

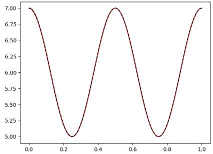
- Finite differences scheme in space and time, avoiding inverse crime : $\neq$ meshes & scheme for direct and inverse problems ;



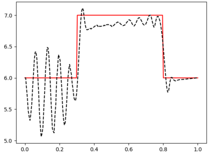
(a) $a = 6 + \sin(2\pi x)$



(b) $a \in \mathcal{V}_{\mathbf{a},\mathbf{a}_\nu}$



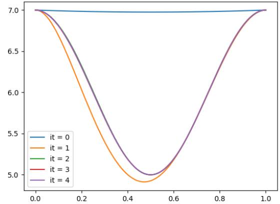
(c) $a \notin \mathcal{V}_{\mathbf{a},\mathbf{a}_\nu}$



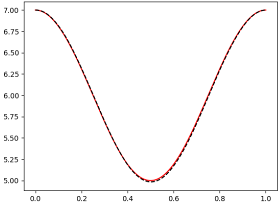
(d) $a \notin \mathcal{V}_{\mathbf{a},\mathbf{a}_\nu}$

Numerical results with noisy data

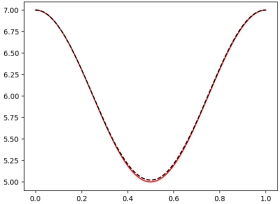
- $\partial_t^2 \mu = (1 + \alpha\,\mathcal{N}(0, 0.5))\partial_t^2 \mu, \quad \alpha \geq 0, \qquad a = 6 + \cos(2\pi x)$



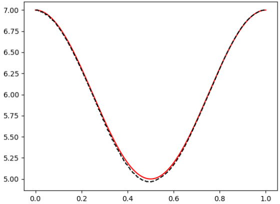
(a) $\alpha = 0\%$ - iterations



(b) $\alpha = 5\%$

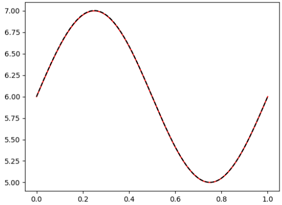


(c) $\alpha = 10\%$

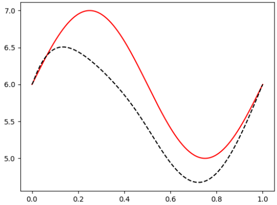


(d) $\alpha = 20\%$

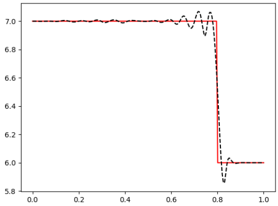
Other numerics



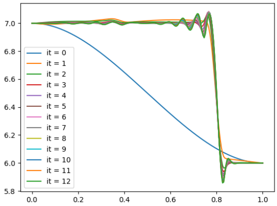
(a) $T = L/\sqrt{\alpha_0}$



(b) $T = L/2\sqrt{\alpha_0}$

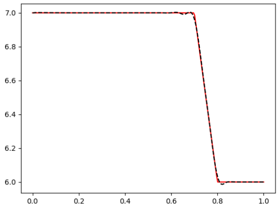


(c) $a \notin \mathcal{V}_{\mathbf{a},\mathbf{a}_\nu}$

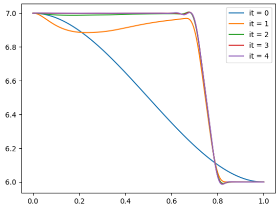


(d) iterations

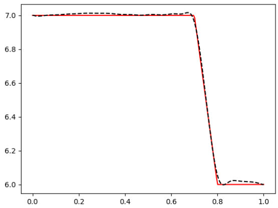
Other numerics



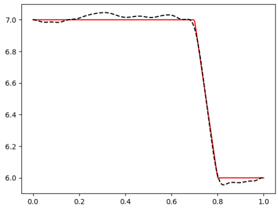
(a) $\alpha = 0\%$



(b) iterations

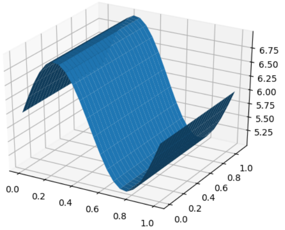


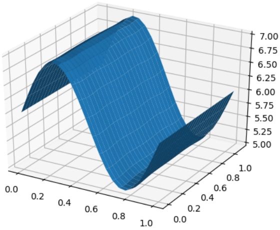
(c) $\alpha = 5\%$



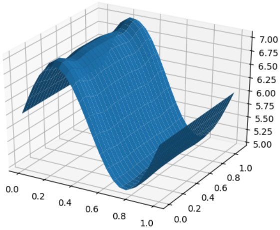
(d) $\alpha = 10\%$

A glimpse to 2D results





(a) Reconstruction with inverse crime



(b) Reconstruction without

⋆ ⋆ ⋆

Conclusion

Flaws

- Projection operator $T_{\mathbf{a},\mathbf{a}_\nu}$ on the admissible set $\mathcal{V}_{\mathbf{a},\mathbf{a}_\nu}$...
- Constraining initial condition on y^0 (inside the domain) :

$$|\nabla y^0 \cdot (x - x_0)| > 0 \text{ in } \Omega.$$

- 2D simulations are not finished *yet* !

Hopes

- Could we design a real imaging system using this strategy ?
- Challenges to work with other equations ?

⋆ ⋆ ⋆

Thank you for your attention.

⋆

Related articles

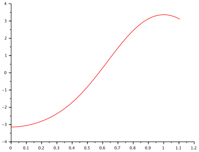
- *Carleman-based Reconstruction algorithm*,
  L. B., M. de Buhan, S. Ervedoza & A. Osses, *in preparation*.
- *Convergent algorithm based on Carleman estimates for the recovery of a potential in the wave equation*,
  L. B., M. de Buhan & S. Ervedoza, SINUM 2017.
- *Stability of an inverse problem for the discrete wave equation and convergence results*,
  L. B., S. Ervedoza & A. Osses, JMPA 2015.
- *Global Carleman estimates for waves and applications*,
  L. B., M. de Buhan & S. Ervedoza, Comm. PDE 2013.
- *Convergence of an inverse problem for discrete wave equations*,
  L. B. & S. Ervedoza, SICON 2013.

Numerics with noise in the data

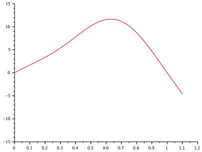
$$\mu = (1 + \alpha\,\mathcal{N}(0, 0.5))\mu, \quad \alpha \geq 0, \qquad a = 6 + \sin(2\pi x)$$

Problem : we derive in time the observations $\partial_t^2 \mu$.

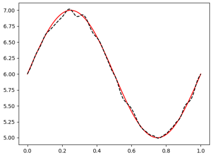
Observation at $x = 1$



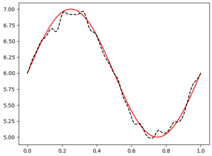
Time derivative



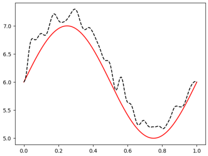
We regularize the signal by convolutions with a gaussian.



(c) $\alpha = 1\%$



(d) $\alpha = 2\%$



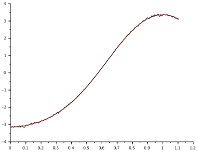
(e) $\alpha = 4\%$

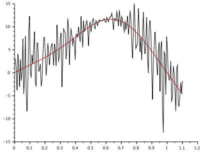
Numerics with noise in the data

$$\mu = (1 + \alpha\,\mathcal{N}(0, 0.5))\mu, \quad \alpha \geq 0, \qquad a = 6 + \sin(2\pi x)$$

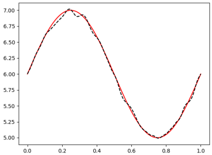
Problem : we derive in time the observations $\partial_t^2 \mu$.

Observation at $x = 1$ Time derivative

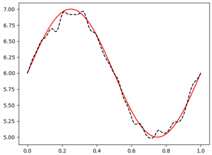




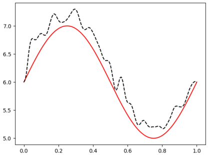
We regularize the signal by convolutions with a gaussian.



(f) $\alpha = 1\%$



(g) $\alpha = 2\%$



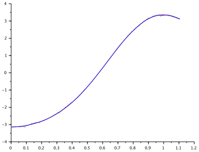
(h) $\alpha = 4\%$

Numerics with noise in the data

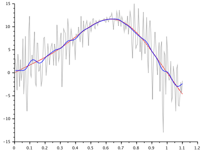
$$\mu = (1 + \alpha \mathcal{N}(0, 0.5))\mu, \quad \alpha \geq 0, \qquad a = 6 + \sin(2\pi x)$$

Problem : we derive in time the observations $\partial_t^2 \mu$.

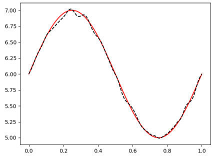
Observation at $x = 1$



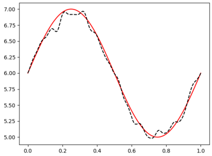
Time derivative



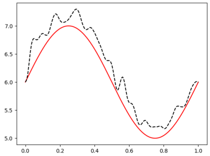
We regularize the signal by convolutions with a gaussian.



(i) $\alpha = 1\%$



(j) $\alpha = 2\%$



(k) $\alpha = 4\%$