Null controllability of linear parabolic-transport systems

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$$\partial_t f - B\partial_x^2 f + A\partial_x f + Kf = u1_\omega\,, \quad \text{in } (0,T)\times\mathbb{T}$$

$$B = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \qquad A = \begin{pmatrix} A' & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \quad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$\Re(\mathrm{Sp}(D)) \subset \mathbb{R}_+^*$

A' diagonalizable

$\mathrm{Sp}(A') \subset \mathbb{R}$

State : $f : (0,T)\times\mathbb{T} \to \mathbb{R}^d$ **Control :** $u : (0,T)\times\mathbb{T}\to\mathbb{R}^d$

$f(t,x) = (f_1,f_2)(t,x) \in \mathbb{R}^{d_1}\times\mathbb{R}^{d_2}\,, \qquad d = d_1 + d_2$

$$\begin{cases} (\partial_t + A'\partial_x + K_{11})\, f_1 + (A_{12}\partial_x + K_{12})\, f_2 = u_1 1_\omega & \text{in } (0,T)\times\mathbb{T} \\ \left(\partial_t - D\partial_x^2 + A_{22}\partial_x + K_{22}\right) f_2 + (A_{21}\partial_x + K_{21})\, f_1 = u_2 1_\omega & \text{in } (0,T)\times\mathbb{T} \end{cases}$$

Two coupled systems : transport of size d_1, parabolic of size d_2.

Well posedness :
$\forall f_0 \in L^2(\mathbb{T})^d, u \in L^2((0,T)\times\mathbb{T})^d, \quad \exists! f \in C^0([0,T], L^2(\mathbb{T})^d)$

Notation : $f(T) = S(T; f_0, u)$

Goal

$$\partial_t f - B\partial_x^2 f + A\partial_x f + Kf = u1_\omega\,, \quad \text{in } (0,T)\times\mathbb{T}$$

$$B = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \qquad A = \begin{pmatrix} A' & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \quad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

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Null controllability : $\forall f_0 \in L^2(\mathbb{T})^d$, $\exists u \in L^2((0,T)\times\mathbb{T})^d \,/\, f(T,.) = 0$.
For which $T > 0$? With less than d controls ? For which f_0 ?

1. Identify the minimal time for null controllability with $u(t,x) \in \mathbb{R}^d$.
2. Use controls $u = (u_1, u_2)$ acting
   - only on the transport component $u = (u_1, 0)$,
   - or only on the parabolic component $u = (0, u_2)$.
3. Understand the influence of the algebraic structure (A, B, K, M) on the null controllability.

Ex 1 & biblio : 1D linearized compressible Navier-Stokes

ρ, v = density, velocity of the fluid, $a, \gamma, \mu > 0$,

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = u_1(t,x) 1_\omega & \text{in } (0,T) \times \mathbb{T} \\ \rho[\partial_t v + v \partial_x v] + \partial_x(a\rho^\gamma) - \mu \partial_x^2 v = u_2(t,x) 1_\omega(x) & \text{in } (0,T) \times \mathbb{T} \end{cases}$$

Linearization around constant stationary state $(\overline{\rho}, \overline{v}) \in \mathbb{R}_+^* \times \mathbb{R}^*$

$$\begin{cases} \partial_t \rho + \overline{v} \partial_x \rho + \overline{\rho} \partial_x v = u_1(t,x) 1_\omega & \text{in } (0,T) \times \mathbb{T} \\ \partial_t v + \overline{v} \partial_x v + a \overline{\rho}^{\gamma-2} \partial_x \rho - \frac{\mu}{\overline{\rho}} \partial_x^2 v = u_2(t,x) 1_\omega(x) & \text{in } (0,T) \times \mathbb{T} \end{cases}$$

Ervedoza, Glass, Guerrero, Puel 2012 : 2 boundary controls on (ρ, v).

Spectral methods : separate variables, $T > \frac{2\pi}{\overline{v}}$
Chowdhury, Mitra, Ramaswamy 2014 : $u = (u_1, u_2)$
Chowdhury, Mitra 2015 : $u = (0, u_2)$, $(\rho_0, v_0) \in H^{7,5^+} \times H^{6,5^+}(\mathbb{T})$
Chowdhury, Mitra, Ramaswamy, Renardy 2014 : $u = (0, u_2)$, $(\rho_0, v_0) \in H^1 \times L^2(\mathbb{T})$ optimal

Ex 2 & biblio : Wave equation with structural damping

$$\partial_t^2 y - \partial_x^2 y - \partial_t \partial_x^2 y + b \partial_t y = u(t,x),$$

This equation splits into a parabolic-transport system by considering $z := \partial_t y - \partial_x^2 y + (b-1)y$,

$$\begin{cases} \partial_t z + z + (1-b)y = u(t,x), \\ \partial_t y - \partial_x^2 y - z + (b-1)y = 0, \end{cases}$$

Rosier, Rouchon 2007 : 1 boundary control on y, obstruction to spectral controllability = accumulation point in the spectrum

Martin, Rosier, Rouchon 2013 : moving control $h = u(t,x) 1_{\omega + ct}$ + CVAR

$$\begin{cases} \partial_t z - c\partial_x z + z + (1-b)y = u(t,x) 1_{\omega}(x) & x \in \mathbb{T} \\ \partial_t y - c\partial_x y - \partial_x^2 y - z + (b-1)y = 0 & x \in \mathbb{T} \end{cases}$$

Moment method : $(y_0, y_1) \in H^{9,5^+} \times H^{7,5^+}(\mathbb{T})$, $T > \frac{2\pi}{c}$

Chaves-Silva, Rosier and Zuazua 2014 : multi-D, $h = u(t,x) 1_{\omega(t)}(x)$
Carleman estimates for PDE and ODE with *the same singular weight*, adapted to the geometry of the moving control support.
→ Fails on $\mathbb{T}$. Guzman, Rosier 2019 : New weight construction on $\mathbb{T}^2$.

$$\begin{cases} \partial_t y - \Delta y - \int_0^t \Delta y(\tau) d\tau = u 1_\omega & (t,x) \in (0,T) \times \Omega \\ y(t,x) = 0 & (t,x) \in (0,T) \times \partial\Omega \end{cases}$$

This equation splits into a parabolic-transport system by considering $v(t,x) = -\int_0^t y_x(\tau) d\tau$:

$$\begin{cases} \partial_t v + \partial_x y = 0 \\ \partial_t y - \partial_x^2 y + \partial_x v = u 1_\omega \\ y(t,0) = y(t,1) = v(t,0) = 0 \end{cases}$$

Ivanov, Pandolfi 2009 : not null controllable 'to the rest'.
Guerrero, Imanuvilov 2013 : not null controllable whatever $T > 0$

Related ex : Linear system of thermoelasticity

$$\begin{cases} \partial_t^2 w - \Delta w + \alpha \Delta \theta = 0, & (t,x) \in (0,T) \times \Omega, \\ \partial_t \theta - \nu \Delta \theta + \beta \partial_t w = 0, & (t,x) \in (0,T) \times \Omega, \\ w(t,x) = u_1(t,x), & (t,x) \in (0,T) \times \partial\Omega, \\ \theta(t,x) = u_2(t,x), & (t,x) \in (0,T) \times \partial\Omega, \end{cases}$$

Albano, Tataru 2000 : boundary control on (w,θ) on $\partial\Omega$, null controllability in large time, Carleman estimate with same weight

Lebeau, Zuazua 1998 : source control on w, null controllability under GCC (Ω, ω, T), **subtle proof relying on a spectral decomposition**.

We will extend this strategy to parabolic-transport systems of any size.

Our 3 results (simplified statement with $K = 0$)

$$\partial_t f - B\partial_x^2 f + A\partial_x f = u 1_\omega \,, \quad \text{in } (0,T)\times\mathbb{T}$$

$$B = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \qquad A = \begin{pmatrix} A' & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$\Re(\mathrm{Sp}(D)) \subset \mathbb{R}_+^*$

A' diagonalizable

$\mathrm{Sp}(A') \subset \mathbb{R}$

[KB, Koenig, Le Balc'h 2019 :]

1. **Minimal time** $\frac{\ell(\mathbb{T}\backslash\omega)}{\mu_*}$ when the control acts on each equation $\mu_* = \min|\mathrm{Sp}(A')|$
2. **NSC with hyperbolic control** $u = (u_1, 0)$ when $D = I_{d_2}$:

   (null controllability in time $T > T_{min}$) $\Leftrightarrow$ (Kalman(A_{22}, A_{21}))
3. **NSC with parabolic control** $u = (0, u_2)$:

   (null controllability in time $T > T_{min}$) $\Leftrightarrow$ (Kalman(A', A_{12}))

For $f_0 = (f_{01}, f_{02}) \in H_m^{d_1+1}(\mathbb{T})^{d_1} \times H^{d_1+1}(\mathbb{T})^{d_2}$.

1. **1D linearized compressible Navier-Stokes :**

$$\begin{cases} \partial_t \rho + \overline{v}\partial_x \rho + \overline{\rho}\partial_x v = u_1(t,x)1_\omega & \text{in } (0,T)\times\mathbb{T} \\ \partial_t v + \overline{v}\partial_x v + a\overline{\rho}^{\gamma-2}\partial_x\rho - \frac{\mu}{\overline{\rho}}\partial_x^2 v = u_2(t,x)1_\omega(x) & \text{in } (0,T)\times\mathbb{T} \end{cases}$$

positive result in optimal time $T_{min} = \frac{\ell(\mathbb{T}\backslash\omega)}{\mu_*}$ (1 or 2 controls)
negative result when $T < T_{min}$

2. **Wave equation with structural damping** :

$$\begin{cases} \partial_t z - c\partial_x z + z + (1-b)y = u(t,x)1_\omega(x) & \text{in } (0,T)\times\mathbb{T} \\ \partial_t y - c\partial_x y - \partial_x^2 y - z + (b-1)y = 0 & \text{in } (0,T)\times\mathbb{T} \end{cases}$$

static control $(c=0)$ *:*
negative result $\forall T>0$ with $u=u(t,x)$ i.e. non separate variables
moving control $(c \neq 0)$ *:*
positive result in optimal time T_{min}
optimal functional spaces : $(y_0, z_0) \in L^2(\mathbb{T})^2$
negative result when $T < T_{min}$

$$B = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \qquad A = \begin{pmatrix} A' & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

$\Re(\mathrm{Sp}(D)) \subset \mathbb{R}_+^*$ $\quad$ A' diagonalizable, $\mathrm{Sp}(A') \subset \mathbb{R}$

If $X \in \mathbb{C}^d$ and $e_n(x) = e^{inx}$ then $\mathcal{L}(Xe_n) = (n^2 B + inA)Xe_n = n^2 E\left(\frac{i}{n}\right) X$ where $E(z) = B + zA$ is an analytic perturbation of B.

Projections on $\mathbb{C}^d$ analytic wrt z :

- $P_\mu^h(z)$ satisfies $E(z)P_\mu^h(z) = \mu z P_\mu^h(z) + z^2 R_\mu^h(z)$ for any $\mu \in \mathrm{Sp}(A')$
- $P^p(z)$ satisfies $E(z)P^p(z) = B + O(z)$
- $P^p(z) + \sum_{\mu \in \mathrm{Sp}(A')} P_\mu^h(z) = I_d$

Key point : $P(z) = \frac{-1}{2i\pi} \int_{\partial\Omega} (E(z) - \zeta I_d)^{-1} d\zeta$ for appropriate Ω

Well posedness in $L^2(\mathbb{T})$: diagonal & uniform bound wrt $n \in \mathbb{Z}$

$$\left| e^{-n^2 E(i/n)t} X \right| \leqslant \left| e^{-(n^2 B + O(n))t} P^p\left(\frac{i}{n}\right) X \right| + \left| e^{(i\mu n + O(1))t} P_\mu^h\left(\frac{i}{n}\right) X \right| \leqslant C|X|$$

$$\partial_t g - B\partial_x^2 g + A\partial_x g = 0\,, \qquad B = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \qquad A = \begin{pmatrix} A' & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

$\Re(\mathrm{Sp}(D)) \subset \mathbb{R}_+^*$

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Construction of a counter-example to the observability inequality

$$\|g(T,.)\|^2_{L^2(\mathbb{T})} \leqslant C \int_0^T \int_\omega |g(t,x)|^2 dxdt$$

Rk : If $\mathrm{rk}(B|...|A^{d-1}B) = d$ then there is no pure transport solution.

Let χ be such that the support of $(t,x) \mapsto \chi(x+\mu t)$ does not intersect $(0,T)\times\omega$. Let $\chi_N(x) := P_N(-id_x)\chi(x) = \sum_{|n|\geqslant N} P_N(n)c_n(\chi)e_n$ where $P_N(X) := \Pi_{-N<j<N}(X-j)$ so that $\mathrm{Supp}(\chi_N) \subset \mathrm{Supp}(\chi)$. Let $X \in \mathrm{Im}[P^h_{\mu_*}(0)]$.

$$\begin{aligned} g_N(t,x) &= \sum_{|n|\geqslant N} P_N(n)c_n(\chi)e^{in(x+\mu_* t)}e^{tR^h_{\mu_*}(i/n)}P^h_{\mu_*}\left(\tfrac{i}{n}\right)X \\ &= \chi_N(x+\mu t)e^{tR^h_\mu(0)}X + \underset{N\to\infty}{O}\left(\tfrac{1}{N}\right) \end{aligned}$$

Theorem 1 : Null controllability in time $T > T_{min}$

$$\partial_t f - B\partial_x^2 f + A\partial_x f = u1_\omega\,, \qquad B = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \qquad A = \begin{pmatrix} A' & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

$\Re(\mathrm{Sp}(D)) \subset \mathbb{R}_+^*$

A' diagonalizable

$\mathrm{Sp}(A') \subset \mathbb{R}$

Theorem 1 : [KB, Koenig, Le Balc'h 2019 :]
Let $T_{min} < T' < T$ and $f_0 \in L^2(\mathbb{T})^d$.
There exists $u = (u_1, u_2) \in L^2((0, T') \times \omega)^{d_1} \times C_c^\infty((T', T) \times \omega)^{d_2}$ such that $f(T, .) = 0$.

Equivalent observability inequality for the adjoint system :

$$\forall s \in \mathbb{N}\,, \exists C_s > 0 / \forall g_0 \in L^2(\mathbb{T})^d \quad \|g(T)\|_{L^2(\mathbb{T})} \leqslant C_s \left[\|g_1\|_{L^2(q_T)} + \|g_2\|_{H^{-s}(q_T)}\right]$$

Rk : The weak H^{-s}-observation will help in eliminating the observation of g_2 to get controllability with $u = (u_1, 0)$ (Theorem 2).

Control strategy

Adapted decomposition : $L^2(\mathbb{T})^d = F^0 \oplus F^p \oplus F^h$ with n_0 'large enough'

$$F^0 = \bigoplus_{|n|\leq n_0} \mathbb{C}^d e_n, \quad F^p = \bigoplus_{|n|>n_0} \mathrm{Im}\left[P^p\left(\frac{i}{n}\right)\right] e_n, \quad F^h = \bigoplus_{|n|>n_0} \mathrm{Im}\left[P^h\left(\frac{i}{n}\right)\right] e_n.$$

Associated (non orthogonal) projections Π^0, Π^p, Π^h, $\Pi = \Pi^p + \Pi^h$
Controls $u = (u_h, u_p) : u_h \in L^2((0,T') \times \omega)^{d_1}$, $u_p \in C_c^\infty((T',T) \times \omega)^{d_2}$.

1. Null controllability on a subspace $\mathcal{G}$ of $L^2(\mathbb{T})^d$ with finite codimension : $\forall f_0 \in \mathcal{G}, \exists u$ such that $\Pi S(T; f_0, u) = 0$.
2. Unique continuation argument (for eigenfunctions) to control the finite dimensional space too.

To prove the first point :

1. $\forall f_0, u_p, \exists u_h$ such that $\Pi^h S(T; f_0, (u_h, u_p)) = 0$
2. $\forall f_0, u_h, \exists u_p$ such that $\Pi^p S(T; f_0, (u_h, u_p)) = 0$
3. Fredholm alternative. Compactness of C_c^∞ in L^2

Control of the hyperbolic high frequencies ($T_{min} < T' < T$)

Goal : $\forall f_0 \in L^2(\mathbb{T})^d, u_p \in L^2((T', T) \times \omega)^{d_2}, \exists u_h \in L^2((0, T') \times \omega)^{d_1}$ such that $\Pi^h S(T; f_0, (u_h, u_p)) = 0$.

1. Reduction to an exact controllability problem :
   $\forall f_{T'} \in F^h, \exists u_h \in L^2((0, T') \times \omega) \,/\quad \Pi^h S(T'; 0, (u_h, 0)) = f_{T'}$
2. Equivalent observability inequality : $\exists C > 0$ such that
   $\forall g_0 \in \widetilde{F^h}, \quad \|g_0\|^2_{L^2(\mathbb{T})^d} \leqslant C \int_0^T \int_\omega |g_1(t,x)|^2 dxdt$
3. For $g_0 \in \widetilde{F^h_\mu}$, the solution of the adjoint system solves a transport equation, 'up to a compact term' $\Rightarrow$ weak observation :
   $\forall g_0 \in \widetilde{F^h_\mu}, \quad \|g_0\|_{L^2(\mathbb{T})^d} \leqslant C \left(\|g_1\|_{L^2(q_T)} + \|g_0\|_{H^{-1}(\mathbb{T})^d} \right)$
4. $+$ unique continuation argument, to remove the H^{-1}-norm
5. $+$ sum to conclude on $\widetilde{F^h}$

Key point : Spectral analysis.

Control of the parabolic high frequencies ($T_{min} < T' < T$)

Goal : $\forall f_0 \in L^2(\mathbb{T})^d, u_h \in L^2((0,T')\times\omega)^{d_1}, \exists u_p \in C_c^\infty((T',T)\times\omega)^{d_2}$ such that $\Pi^p S(T; f_0, (u_h, u_p)) = 0$.

1. Reduction to a null controllability pb :
   $\forall \tau > 0\,, f_0 \in F^p, \exists u_p \in C_c^\infty((0,\tau)\times\omega)^{d_2} / \quad \Pi^p S(\tau; f_0, (0, u_p)) = 0$
   Important choice of the control supports : $\tau = T - T'$
2. Equivalent observability inequality
   $\forall g_0 \in \widetilde{F^p}\,, \quad \|g_0\|^2_{L^2(\mathbb{T})^d} \leqslant C \int_0^T \int_\omega |g_2(t,x)|^2 dxdt$
3. **Equation satisfied by the parabolic components :**
   For z small enough, $X \in \mathrm{Im}[P^p(z)] \Leftrightarrow X_1 = G(z)X_2$
   For $g_0 \in \widetilde{F^p}$, $g_1 = Gg_2 = \sum_{|n|>n_0} G\left(\frac{i}{n}\right) c_n(g_2) e_n$
   $(\partial_t - D\partial_x^2 - A_{22}\partial_x)g_2 - A_{21}\partial_x[Gg_2] = 0$
4. Lebeau-Robbiano's method, **'by bloc'**, produces **smooth controls** $u_p \in C_c^\infty$.

Simpler proof with zero order coupling.

$$\begin{cases} (\partial_t + A'\partial_x + K_{11}) f_1 + (A_{12}\partial_x + K_{12}) f_2 = u_1 1_\omega & \text{in } (0,T)\times\mathbb{T} \\ (\partial_t - \partial_x^2 + K_{22}) f_2 + K_{21} f_1 = 0 & \text{in } (0,T)\times\mathbb{T} \end{cases}$$

Theorem 2 : [KB, Koenig, Le Balc'h 2019 :]
Null controllability in time $T > T_{min} \Leftrightarrow$ Kalman condition

$$\text{Span}\{{K_{22}}^j K_{21} X_1 ; X_1 \in \mathbb{R}^{d_1}, 0 \leq j \leq d_2 - 1\} = \mathbb{R}^{d_2}$$

Necessary : $X_2'(t) + (n^2 I_{d_2} + K_{22}) X_2(t) + K_{21} X_1(t) = 0$ must be controllable with state X_2 and control X_1.

Sufficient : We know that $\|g(T)\|_{L^2} \leqslant C\left[\|g_1\|_{L^2(q_T)} + \|g_2\|_{H^{-2d_2+1}(q_T)}\right]$.
Under Kalman condition, a bloc structure allows to prove
$\|g_2\|_{H^{-2d_2+1}(q_T)} \leqslant C\|g_1\|_{L^2(q_T)}$

Elimination of $\|g_2\|_{H^{-2d_2+1}(q_T)}$ when $d_1 = 1$

Up to a change of basis on $\mathbb{R}^{d_2}$

$$K_{22} = \begin{pmatrix} 0 & \dots & \dots & 0 & c_0 \\ 1 & 0 & \dots & \vdots & c_1 \\ 0 & \ddots & \ddots & \vdots & c_2 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \dots & 0 & 1 & c_{d_2-1} \end{pmatrix} \text{ and } K_{21} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Thus, in the adjoint system

$$\begin{cases} \left(\partial_t - A'^{\mathrm{tr}}\partial_x + K_{11}^{\mathrm{tr}}\right) g_1 + K_{21}{}^{\mathrm{tr}} g_2 = 0 \\ \left(\partial_t - \partial_x^2 + K_{22}{}^{\mathrm{tr}}\right) g_2 + \left(-(A_{12}P)^{\mathrm{tr}}\partial_x + (K_{12}P)^{\mathrm{tr}}\right) g_1 = 0 \end{cases}$$

for every $i \in \{2, \dots, d_2\}$, the i-th equation is

$$(\partial_t - \partial_x^2) g_2^{i-1} + g_2^i + b_{i-1}\partial_x g_1 + a_{i-1} g_1 = 0$$

thus $\quad \|g_2^i\|_{H^{-2i+1}(q_T)} \leq C\left(\|g_2^{i-1}\|_{H^{-2(i-1)+1}(q_T)} + \|g_1\|_{L^2(q_T)}\right)$

and the first equation $\quad (\partial_t - A'\partial_x + K_{11}) g_1 + g_2^1 = 0$

gives $\quad \|g_2^1\|_{H^{-1}(q_T)} \leq C\|g_1\|_{L^2(q_T)}$.

Theorem 2 : Hyperbolic control, coupling of order one

$$\begin{cases} (\partial_t + A'\partial_x + K_{11}) f_1 + (A_{12}\partial_x + K_{12}) f_2 = u_1 1_\omega \\ (\partial_t - \partial_x^2 + A_{22}\partial_x) f_2 + A_{21}\partial_x f_1 = 0 \end{cases}$$

Theorem 2' : [KB, Koenig, Le Balc'h 2019 :]
Null controllability in time $T > T_{min} \Leftrightarrow$ Kalman condition

$$\text{Span}\{{A_{22}}^j A_{21} X_1 ; X_1 \in \mathbb{R}^{d_1}, 0 \leq j \leq d_2 - 1\} = \mathbb{R}^{d_2}$$

A variation of the proof of Theorem 1 (with $u_p \leftarrow \partial_x^{d_2} u_p$) gives the following observability inequality for the adjoint system

$$\|g(T)\|_{L^2(\mathbb{T})^d} \leqslant C \left(\|g_1\|_{L^2(q_T)} + \|\partial_x^{d_2} g_2\|_{H^{-2d_2+1}} \right).$$

Then, for $i \in \{2, \ldots, d_2\}$, by applying ∂_x^{i-1} to the i-th equation

$$(\partial_t - \partial_x^2) g_2^{i-1} + \partial_x g_2^i + b_{i-1}\partial_x g_1 + a_{i-1} g_1 = 0$$

we get $\|\partial_x^i g_2^i\|_{H^{-2i+1}(q_T)} \leq C \left(\|\partial_x^{i-1} g_2^{i-1}\|_{H^{-2(i-1)+1}(q_T)} + \|g_1\|_{L^2(q_T)} \right).$

Theorem 3 : Parabolic control

$$\begin{cases} (\partial_t + A'\partial_x)\, f_1 + A_{12}\partial_x f_2 = 0 \\ \left(\partial_t - D\partial_x^2 + A_{22}\partial_x + K_{22}\right) f_2 + (A_{21}\partial_x + K_{21}) f_1 = u_2 1_\omega \end{cases}$$

Theorem 3 : [KB, Koenig, Le Balc'h 2019 :]
Null controllability in time $T > T_{min}$ of any $(f_1^0, f_2^0) \in H_m^{d_1+1}(\mathbb{T})^{d_1} \times H^{d_1+1}(\mathbb{T})$ is equivalent to the Kalman condition

$$\text{Span}\{(A')^j A_{12} X_2; X_2 \in \mathbb{C}^{d_2}, 0 \leq j \leq d_2 - 1\} = \mathbb{C}^{d_2}$$

Rk : A regularity assumption is necessary.

Ex : $\begin{cases} (\partial_t + c\partial_x) f_1 + \partial_x f_2 = 0, & \text{in } (0,T) \times \mathbb{T}, \\ (\partial_t - \partial_x^2 + c\partial_x) f_2 = v(t,x), & \text{in } (0,T) \times \mathbb{T}. \end{cases}$

An initial condition $f^0 = (f_1^0, f_2^0) \in L_m^2(\mathbb{T}) \times L^2(\mathbb{T})$ is null controllable with $v \in L^2((0,T) \times \mathbb{T})$ if and only if $f_{01} \in H^1(\mathbb{T})$.

Proof :

1. Equivalent observability inequality :

$$\|g(T)\|^2_{H^{-(d_1+1)}(\mathbb{T})^d} \leqslant C \int_0^T \int_\omega |g_2(t,x)|^2 dxdt$$

2. By the same elimination strategy, it suffices to prove

$$\|g(T)\|_{H^{-(d_1+1)}(\mathbb{T})^d} \leqslant C \left[\|\partial_x^{d_1} g_1\|_{H^{-(d_1+1)}(\mathbb{T})^d} + \|g_2\|_{L^2(q_T)} \right]$$

   $(d_1 - 1)$ eliminations cost 1 derivative + 1 cost 2 derivatives

3. We need to control the H^{d_1+1}-hyperbolic dynamics with controls $\partial_x^{d_1} u_h$ where $u_h \in H_0^{2d_1+1}((0,T') \times \omega)$. This is ensured by the control construction in [Alabau, Coron, Olive 2017].

Progress, open problems and perspectives

What is new :

- Extension of the Lebeau-Zuazua's strategy to systems of any size.
- We don't need eigenvalues/eigenvectors, only projections on generalized eigenspaces. Kato's perturbation theory.
- The 'bloc' Lebeau-Robbiano's method produces arbitrary smooth controls.
- Negative results in $T < T_{min}$ with $u = u(t, x)$.

Open problems :

- Optimal regularity assumption on f_0 when $u = (0, u_p)$?
- With less controls ?
- Unique continuation in small time ?
- Higher dimension ?
- Non constant coefficients ?