Bilinear control for evolution equations of parabolic type

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CONTROL AND STABILIZATION ISSUES FOR PDEs
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in honor of Jean-Pierre Raymond

Control systems

In a given Banach space X

Dynamical system: $$u' = f(u, \mathbf{p})$$

$\uparrow$ **control function**

where

- $u : [0, T] \to X$ is the state variable
- $\mathbf{p}$ is the control

Additive control for linear systems

$$\begin{cases} u'(t) + Au(t) + B\mathbf{p}(t) = 0 & t \in [0, T] \\ u(0) = u_0 \end{cases}$$

where

- $A : D(A) \subset X \to X$ is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators on X
- $B : D(B) \subset X \to X$ is a linear operator on X that can be either bounded (locally distributed control) or unbounded (boundary control)
- $\mathbf{p} : [0, T] \to X$ is the control

Bilinear control system

$$\begin{cases} u'(t) + Au(t) + \mathbf{p}(t)Bu(t) = 0 & t \in [0, T] \\ u(0) = u_0 \end{cases}$$

In this talk:

- the state space $(X, \langle \cdot, \cdot \rangle)$ is a Hilbert space
- $B : X \to X$ is a bounded linear operator
- $\mathbf{p} : [0, T] \to \mathbb{R}$ is scalar function

What are the difficulties?

The map $\boldsymbol{\Phi} : \mathbf{p} \mapsto u$ is

Additive control:

$$\begin{cases} u' + Au + B\mathbf{p} = 0 \\ u(0) = u_0 \end{cases}$$

Bilinear control:

$$\begin{cases} u' + Au + \mathbf{p}Bu = 0 \\ u(0) = u_0 \end{cases}$$

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Additive control:

$$\begin{cases} u' + Au + B\mathbf{p} = 0 \\ u(0) = u_0 \end{cases}$$

↓

linear

Bilinear control:

$$\begin{cases} u' + Au + \mathbf{p}Bu = 0 \\ u(0) = u_0 \end{cases}$$

↓

nonlinear

An obstruction to exact controllability

Bilinear control:

$$\begin{cases} u' + Au + \mathbf{p}Bu = 0 \\ u(0) = u_0 \end{cases} \tag{1}$$

Let $u_0 \in X$ and denote by $u(\cdot; p, u_0)$ the unique solution of (1) for $p \in L^1_{loc}(0, \infty)$.

Theorem (Ball, Marsden, Slemrod 1982)

If $\dim X = \infty$, *then the attainable set from* u_0

$$S(u_0) = \left\{u(t; p, u_0); t \geq 0, p \in L^1_{loc}(0, \infty)\right\}$$

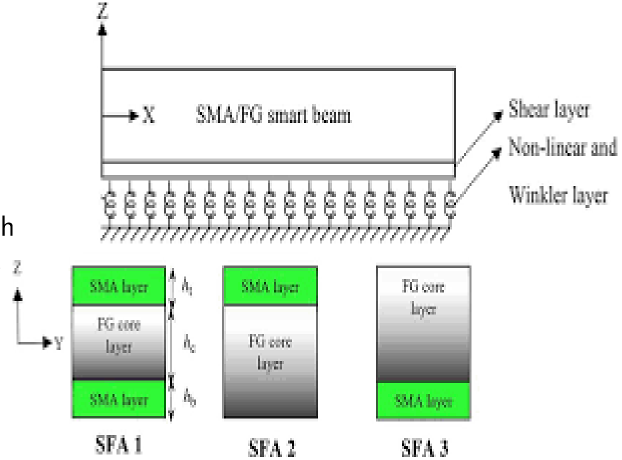
is contained in a countable union of compact subsets of X. *So,* $X \setminus S(u_0)$ *is dense.*

Motivations

Bilinear controls enter the system equations as coefficients chenging (at least some of) the principal parameters of the process at hand

Examples

- by embedded *smart* alloys, the natural frequency response of a beam can be changed
- the rate of a chemical reaction can be altered by various catalysts and/or by th speed at which the reaction ingredients are mechanically mixed

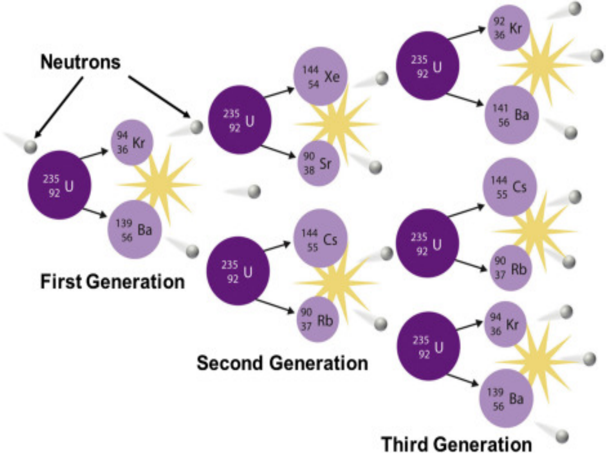


A simplified model of a nuclear chain reaction

A chain reaction refers to a process in which neutrons released in fission produce an additional fission in at least one further nucleus. This nucleus in turn produces neutrons, and the process repeats. The process may be controlled (nuclear power) or uncontrolled (nuclear weapons).

$$\boxed{u_t = a^2 \Delta u + v(t,x) u}$$

- $u(t,x) \geq 0$ neutron density in the reaction
- $v(t,x) > 0$ neutron amount in the surrounding medium
- $v(t,x)u$ neutrons provided by the collision of the particles in the reaction with the surrounding medium

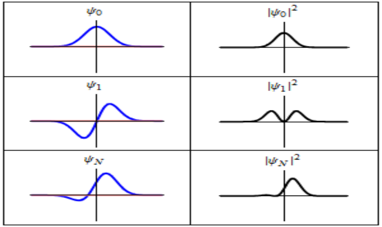


Schrödinger equation

The Schrödinger equation is a linear partial differential equation that describes the wave function or state function of a quantum-mechanical system

$$\boxed{i\psi_t = -\Delta\psi - p(t)\mu(x)\psi}$$

- ψ wave function of a particle
- p amplitude of the electric field
- μ dipolar moment of the particle



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A look at stabilization

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Figure: Jean-Pierre inspired by James Stewart

Notions of stabilizability

$$\begin{cases} u' + Au + \mathbf{p}Bu = 0 \quad (t > 0) \\ u(0) = u_0 \end{cases} \tag{2}$$

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Let $\bar{p} \in L^1_{loc}(0, \infty)$ and let $\bar{u}_0 \in X$

Definitions

- (2) is *locally stabilizable to $u(\cdot; \bar{u}_0, \bar{p})$* if $\exists\, \delta > 0$ such that for all $u_0 \in B_\delta(\bar{u}_0)$ there exists $p \in L^1_{loc}(0, \infty)$ such that
$$\lim_{t \to +\infty} ||u(t; u_0, p) - u(t; \bar{u}_0, \bar{p})|| = 0$$

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- (2) is *locally exponentially stabilizable to $u(\cdot;\bar{u}_0,\bar{p})$* if $\exists\,M,\delta,\rho > 0$ such that for all $u_0 \in B_\delta(\bar{u}_0)$ there exist $p \in L^1_{loc}(0,\infty)$ satisfying
$$||u(t;u_0,p) - u(t;\bar{u}_0,\bar{p})|| \le Me^{-\rho t} \qquad \forall t > 0$$

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$$||u(t; u_0, p) - u(t; \bar{u}_0, \bar{p})|| \leq Me^{-\rho t} \qquad \forall t > 0$$
- (2) is *locally superexponentially stabilizable to $u(\cdot; \bar{u}_0, \bar{p})$* if for any $\rho > 0$ there exists $\delta > 0$ such that, $\forall\, u_0 \in B_\delta(\bar{u}_0)$, it holds that
$$||u(t; u_0, p) - \bar{u}(t; \bar{u}_0, \bar{p})|| \leq Me^{-\rho e^{\omega t}} \qquad \forall t > 0$$

  for some constants $\exists\, M, \omega > 0$

Assumptions

Let $(X, \langle \cdot, \cdot \rangle)$ be a separable Hilbert space and $A : D(A) \subset X \to X$ a densely defined linear operator with the following properties:

$$
\begin{array}{ll}
(a) & A \text{ is self-adjoint} \\
(b) & \langle Ax, x \rangle \geq 0 \qquad \forall x \in D(A) \\
(c) & D(A) \subseteq X \text{ is compact}
\end{array}
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$$\downarrow$$

1. X has an orthonormal basis $\{\varphi_k\}_{k \in \mathbb{N}^*}$ of eigenvectors of $\boldsymbol{A}$
2. the eigenvalues $\{\lambda_k\}_{k \in \mathbb{N}^*}$ of $\boldsymbol{A}$ are nonnegative and $\lambda_k \to +\infty$ as $k \to +\infty$
3. $-\boldsymbol{A}$ generates a strongly continuous semigroup of contractions $e^{-t\boldsymbol{A}}$

Preliminaries

Given $T > 0$, consider the bilinear control problem

$$\begin{cases} u'(t) + Au(t) + p(t)Bu(t) = 0, & t \in [0, T] \\ u(0) = u_0 \end{cases} \tag{4}$$

where $B \in \mathcal{L}(X)$ and $p \in L^2(0, T)$

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Consider system (4) with $p = 0$:

$$\begin{cases} u'(t) + Au(t) = 0, & t \in [0, T] \\ u(0) = \varphi_1. \end{cases}$$

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Remark

If $\langle Ax, x\rangle \geq \nu|x|^2$ $(\nu > 0)$, *then* $p = 0$ *yields*

$$||u(t) - \psi_1(t)|| = ||e^{-tA}u_0 - e^{-tA}\varphi_1|| \leq e^{-\nu t}||u_0 - \varphi_1||$$

So, (4) is locally exponentially (but not superexponentially) stabilizable to ψ_1

Superexponential stabilizability

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Theorem

Suppose there exists a constant $\gamma > 0$ such that the eigenvalues of A fulfill the gap condition

$$\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} \geq \gamma \qquad \forall k \geq 1$$

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Let $B : X \to X$ *be a linear bounded operator with the following properties:*

$$\langle B\varphi_1, \varphi_k \rangle \neq 0 \quad \forall k \geq 1 \qquad \& \qquad \exists \tau > 0 \quad \text{such that} \quad \sum_{k=1}^{\infty} \frac{e^{-2\lambda_k \tau}}{|\langle B\varphi_1, \varphi_k \rangle|^2} < \infty \qquad (\star)$$

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Then, $\forall\, \rho > 0$, $\exists\, R > 0$ such that any $u_0 \in B_R(\varphi_1)$ admits a control $p \in L^2_{loc}(0, \infty)$ such that

$$||u(t) - \psi_1(t)|| \leq M e^{-\rho e^{\omega t} - \lambda_1 t} \qquad \forall t \geq 0$$

where M and ω are positive constants depending only on A and B

Sketch of the proof, $\lambda_1 = 0$

Fix $T > \tau$ where $\tau > 0$ is given by $(\star)$

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$$v := u - \psi_1$$

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$$t \in [0, T] \qquad (\text{where } T > \tau)$$

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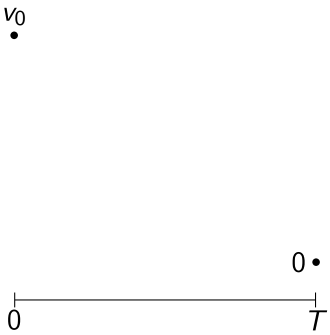

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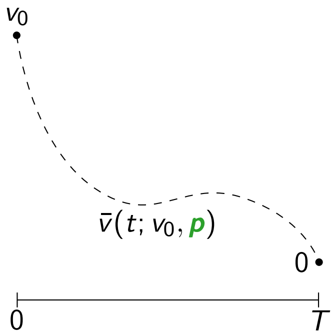

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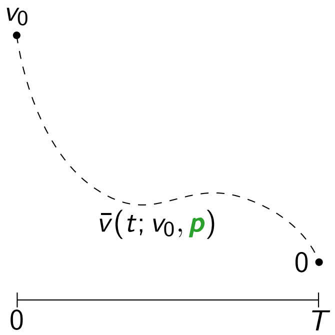

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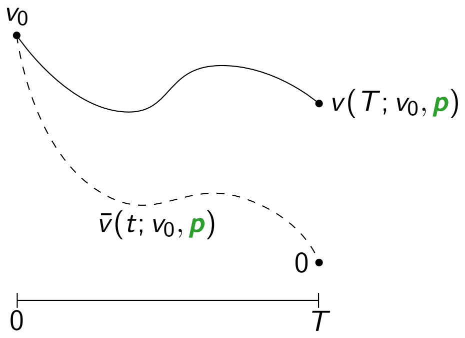

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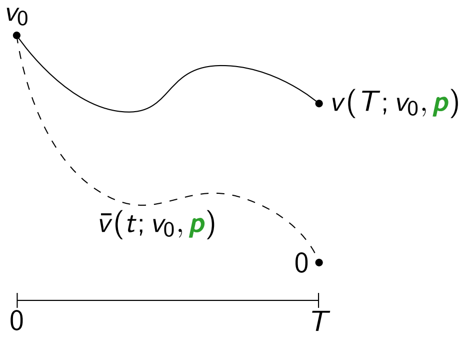

$$\|p\|_{L^2(0,T)} \le \Lambda_T \|v_0\| \qquad \|(v - \bar{v})(T)\| = \|v(T)\| \le K_T \|v_0\|^2.$$

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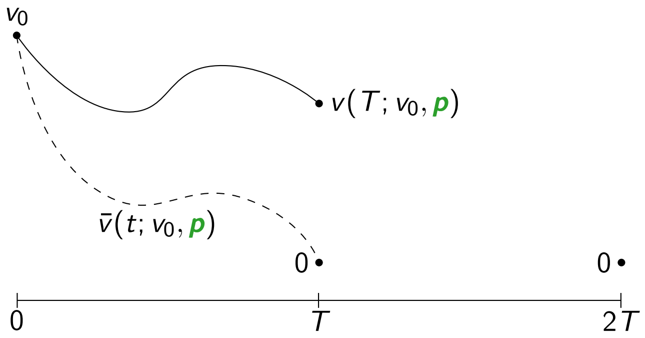

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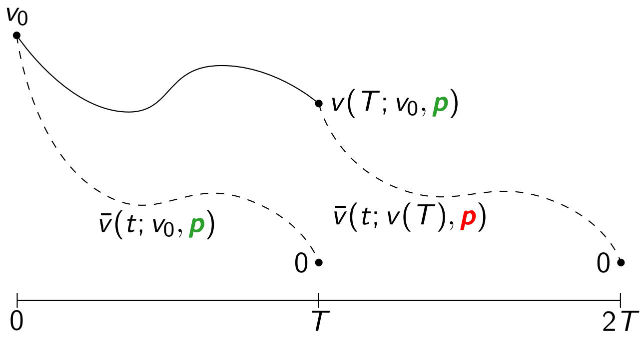

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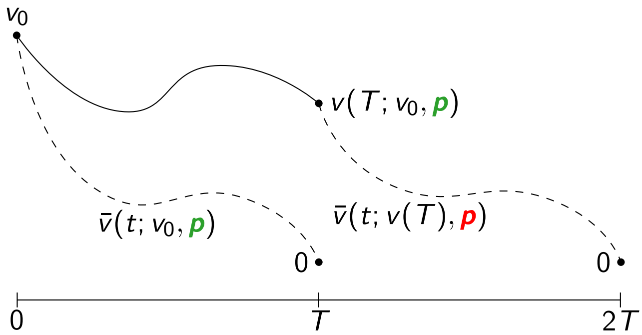

$$||p||_{L^2(T,2T)} \leq \Lambda_T ||v(T)||$$

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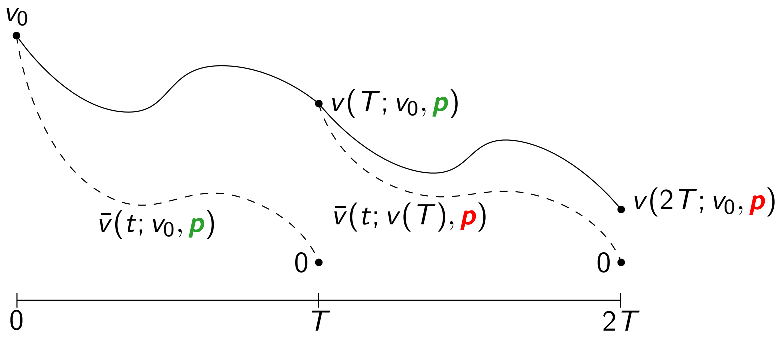

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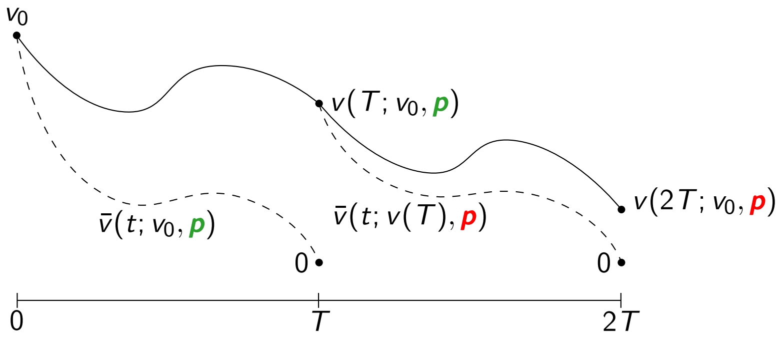

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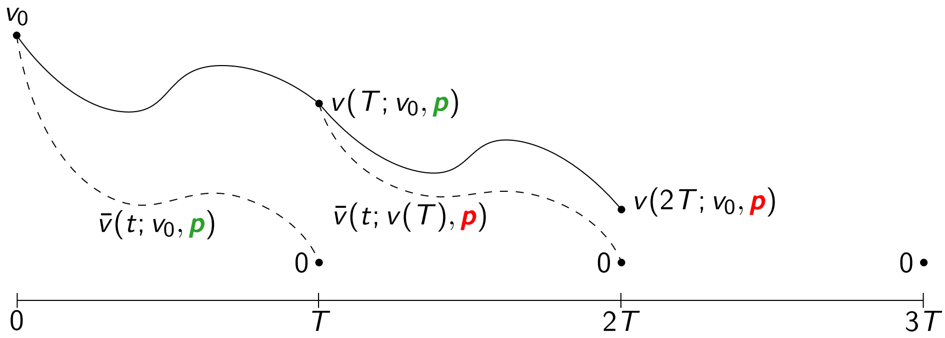

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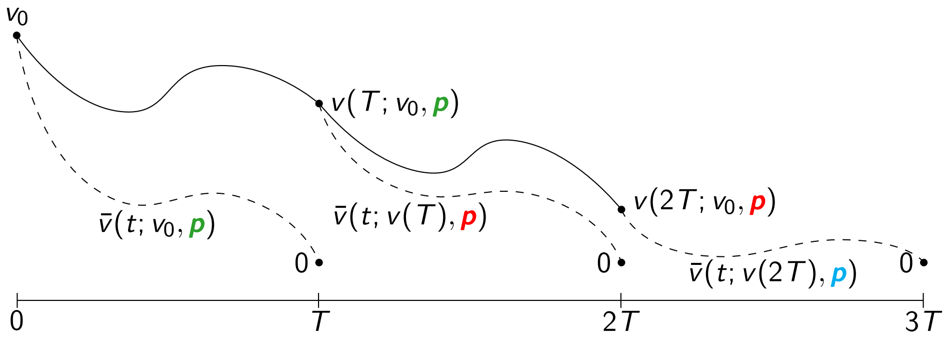

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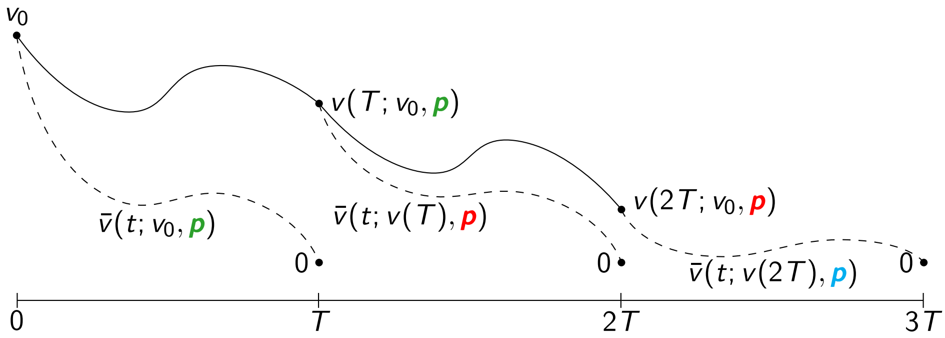

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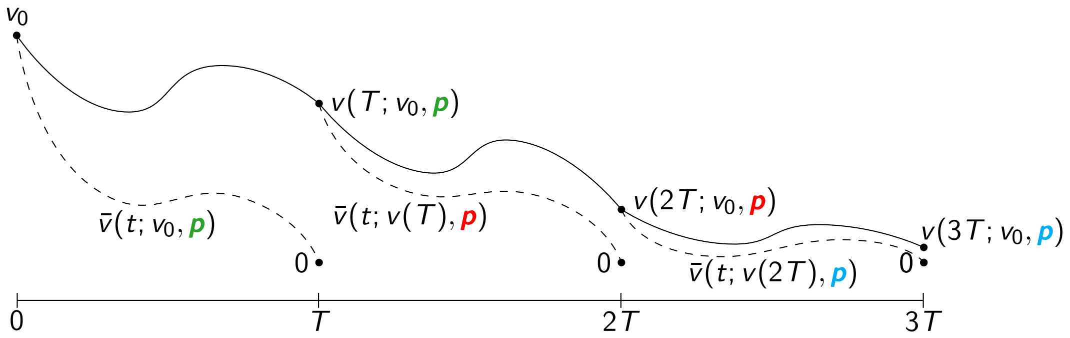

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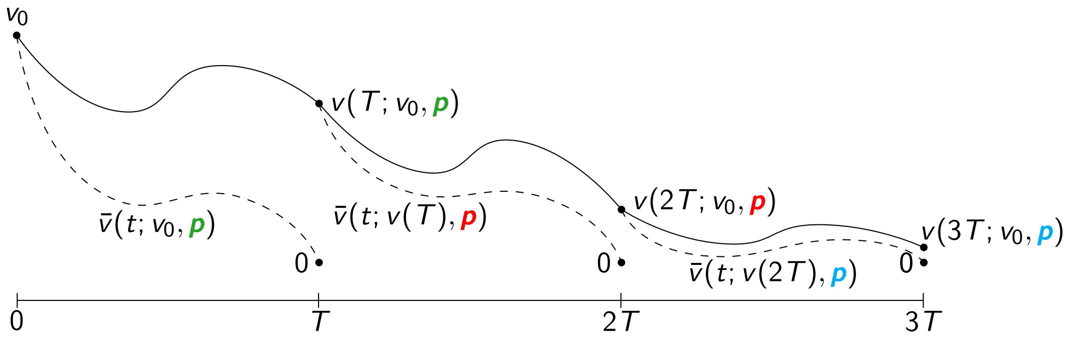

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$$\begin{cases} v'(t) + Av(t) + p(t)Bv(t) + p(t)B\psi_1(t) = 0 \\ v(0) = v_0 = u_0 - \varphi_1 \end{cases} \qquad \begin{cases} \bar{v}(t)' + A\bar{v}(t) + p(t)B\psi_1(t) = 0 \\ \bar{v}(0) = v_0 \end{cases}$$

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where $\{\sigma_k\}_{k\geq 1}$ is biorthogonal to $\{e^{\lambda_k s}\}_{k\geq 1}$.

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Sketch of the proof, $\lambda_1 = 0$

- $$||v(nT)|| \le \frac{1}{K_T}(K_T||v_0||)^{2^n} \qquad \forall n \ge 0 \tag{8}$$
- In any time interval $[nT, (n+1)T]$ we prove that
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- Let $\theta \in (0,1)$ and let $\rho > 0$ be the value for which $\theta = e^{-2\rho}$. Rhen $\exists\ R_\rho > 0$ such that, for all $||u_0 - \varphi_1|| \leq R_\rho$, we have

$$||u(t) - \varphi_1|| \leq M_T e^{-\rho e^{\omega_T t}} \quad \forall t \geq 0$$

with $M_T, \omega_T > 0$ suitable constants

Sketch of the proof, $\lambda_1 > 0$

We introduce the operator

$$A_1 := \boldsymbol{A} - \boldsymbol{\lambda_1} I.$$

Notice that $A_1 : D(A_1) \subset X \to X$ is self-adjoint, accretive and $-A_1$ generates a strongly continuous semigroup of contractions. Its eigenvalues are given by

$$\mu_k = \boldsymbol{\lambda_k} - \boldsymbol{\lambda_1}, \qquad \forall k \in \mathbb{N}^*$$

(in particular, $\mu_1 = 0$) and A_1 has the same eigenfunctions as $\boldsymbol{A}$, $\{\boldsymbol{\varphi_k}\}_{k \geq 1}$. Moreover, the family $\{\mu_k\}_{k \geq 1}$ satisfies the same gap condition that is satisfied by the eigenvalues of $\boldsymbol{A}$

Sketch of the proof, $\lambda_1 > 0$

We define $z(t) = e^{\boldsymbol{\lambda_1} t} u(t)$. Then

$$\begin{cases} z'(t) + A_1 z(t) + p(t)\boldsymbol{B} z(t) = 0 \quad (t > 0) \\ z(0) = u_0. \end{cases}$$

So, we can apply the previous analysis to this problem:

$$||z(t) - \boldsymbol{\varphi_1}|| \leq M_T e^{-\rho e^{\omega_T t}} \qquad \forall t \geq 0$$

Therefore, returning to u,

$$||u(t) - \psi_1(t)|| = ||e^{-\boldsymbol{\lambda_1} t} z(t) - e^{-\boldsymbol{\lambda_1} t}\boldsymbol{\varphi_1}|| = e^{-\boldsymbol{\lambda_1} t}||z(t) - \boldsymbol{\varphi_1}|| \leq M_T e^{-(\rho e^{\omega_T t} + \boldsymbol{\lambda_1} t)} \quad \forall t \geq 0$$

Example 1 (heat eqn, Dirichlet bc)

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Consider the bilinear control system

$$\begin{cases} u_t(t,x) - u_{xx}(t,x) + p(t)\mu(x)u(t,x) = 0, & (t,x) \in (0,\infty) \times (0,1) \\ u(t,0) = u(t,1) = 0 \\ u(0,x) = u_0(x) \end{cases}$$

Example 1

Let $X = L^2(0,1)$ and consider the bilinear control system

$$\begin{cases} u_t(t) + Au(t) + p(t)Bu(t) = 0, & t \in (0,\infty) \\ u(0) = u_0(x) \end{cases} \tag{10}$$

where A and B are defined by

$$\begin{array}{ll} D(A) = H^2 \cap H_0^1(0,1), & A\varphi = -\frac{d^2\varphi}{dx^2} \\ B \in \mathcal{L}(X), & B\varphi = \mu\varphi \end{array} \tag{11}$$

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We want to study the superexponential stabilizability of (27)-(27) to the ground state solution $\psi_1 = e^{-\lambda_1 t}\varphi_1$

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$$|\langle B\varphi_1, \varphi_k\rangle| \geq C{\lambda_k}^{-3/2}, \quad \forall k \in \mathbb{N}^*.$$

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- The series

$$\sum_{k\in\mathbb{N}^*} \frac{e^{-2\lambda_k \tau}}{|\langle B\varphi_1, \varphi_k\rangle|^2}$$

converges for all $\tau > 0$.

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Example 2

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is locally superexponentially stabilizable to $\psi_0(t) = \varphi_0$

Example 3 (degenerate heat eqn)

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Let $0 \le \alpha < 2$ and consider the bilinear control system

$$\begin{cases} u_t - (x^\alpha u_x)_x + p(t)\mu(x)u = 0, & (t,x) \in (0,\infty)\times(0,1) \\ u(t,1) = 0, \quad \begin{cases} u(t,0) = 0, & \text{if } \alpha \in [0,1), \\ (x^\alpha u_x)(t,0) = 0, & \text{if } \alpha \in [1,2) \end{cases} \\ u(0,x) = u_0(x) \end{cases}$$

Example 3: weakly degenerate case ($\alpha < 1$)

Let $X = L^2(0,1)$ and consider the bilinear control system

$$\begin{cases} u_t(t) + \mathbf{A}u(t) + p(t)\mathbf{B}u(t) = 0, & t \in (0,\infty) \\ u(0) = u_0(x). \end{cases}$$

where $\mathbf{A}$ and $\mathbf{B}$ are defined by

$$\begin{array}{ll} D(\mathbf{A}) = \left\{ u \in H^1_{\alpha,0}(0,1) \ : \ x^\alpha \frac{du}{dx} \in H^1(0,1) \right\}, & \mathbf{A}\varphi = -\frac{d}{dx}(x^\alpha \frac{du}{dx}) \\ \mathbf{B} \in \mathcal{L}(X), & \mathbf{B}\varphi = \mu\varphi \end{array}$$

where

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Then

- A is a **self-adjoint accretive** operator and $-A$ generates a C^0-semigroup of contractions

Example 3: gap condition ($\alpha < 1$)

For any $\nu \geq 0$, we denote by

- J_ν the Bessel function of the first kind and order ν
- $j_{\nu,1} < j_{\nu,2} < \cdots < j_{\nu,k} < \ldots$ the sequence of all positive zeros of J_ν

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Then eigenvalues and eigenvectors of A are given by :

$$\lambda_{\alpha,k} = k_\alpha^2 j_{\alpha,k}^2 \qquad (k \geq 1)$$

$$\varphi_{\alpha,k}(x) = \frac{\sqrt{2k_\alpha}}{|J'_{\nu_\alpha}(j_{\nu_\alpha,k})|} x^{(1-\alpha)/2} J_{\nu_\alpha}\left(j_{\nu_\alpha,k} x^{k_\alpha}\right) \qquad (k \geq 1)$$

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So, the gap condition is satisfied :

$$\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} = k_\alpha \left(j_{\nu_\alpha,k+1} - j_{\nu_\alpha,k}\right) \geq k_\alpha \left(j_{\nu_\alpha,2} - j_{\nu_\alpha,1}\right) \geq \frac{7}{16}\pi$$

Example 3: conclusion

- Taking $\boxed{\mu(x) = x^{2-\alpha}}$ a long computation shows that

$$|\langle \mu\varphi_1, \varphi_k\rangle| \geq \frac{C}{\lambda_k^{3/2}} \quad \forall k \geq 1$$

  for some $C > 0$

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$$\begin{cases} u_t(t,x) - u_{xx}(t,x) + p(t)x^{2-\alpha}u(t,x) = 0, & (t,x) \in (0,\infty) \times \Omega \\ u_x(t,0) = u_x(t,1) \\ u(0,x) = u_0(x) \end{cases}$$

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$$\langle B\varphi_1, \varphi_1 \rangle \neq 0 \qquad \& \qquad \exists\, b, q > 0 \quad \text{such that} \quad \lambda_k^q |\langle B\varphi_1, \varphi_k \rangle| \geq b \quad \forall k > 1 \qquad (\star\star)$$

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Then for any $T > 0$ *there exists* $R_T > 0$ *such that, for any* $u_0 \in B_{R_T}(\varphi_1)$*, the solution to* (S) *can be steered to the ground state solution in time* T *by some control* $p \in L^2(0, T)$

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Notice that $(\star\star)$ is satisfied by all the above examples. Moreover

$$(\star\star) \qquad \Longrightarrow \qquad \sum_{k=1}^{\infty} \frac{e^{-2\lambda_k \tau}}{|\langle B\varphi_1, \varphi_k \rangle|^2} < \infty \quad \forall \tau > 0$$

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Then there exists $r_1 > 0$ *such that for all* $0 < r < r_1$ *and all* $R > 0$ *there exists* $T_{r,R} > 0$ *such that for all* $u_0 \in X$ *in the strip*

$$|\langle u_0, \varphi_1 \rangle - 1| \leq r,$$
$$||u_0 - \langle u_0, \varphi_1 \rangle \varphi_1|| \leq R,$$

the solution to (S) *can be steered to the ground state solution* $\psi_1(t) = e^{-\lambda_1 t}\varphi_1$ *in time* $T_{r,R}$ *by some control* $p \in L^2(0, T_{r,R})$

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If $\langle u_0, \varphi_1 \rangle \neq 0$, then for every $R > 0$ there exists $T_R > 0$ such that for all u_0 satisfying

$$||u_0 - \langle u_0, \varphi_1 \rangle \varphi_1|| \leq R |\langle u_0, \varphi_1 \rangle|$$

the solution to (S) can be steered to $\langle u_0, \varphi_1 \rangle \psi_1$ in time T_R some control $p \in L^2(0, T_R)$

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for the eigenvalues of A, we have shown that the control system

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This approach can be extended to suitable classes of (unbounded) operators $B : D(B) \subset X \to X$

Best wishes to Jean-Pierre!

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Thank you