Bilinear control for evolution equations of parabolic type

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CONTROL AND STABILIZATION ISSUES FOR PDEs
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in honor of Jean-Pierre Raymond
Control systems

In a given Banach space $X$

Dynamical system: $u' = f(u, p)\uparrow$

\text{control function}

where

- $u : [0, T] \rightarrow X$ is the state variable
- $p$ is the control
Additive control for linear systems

\[
\begin{aligned}
  u'(t) + Au(t) + Bp(t) &= 0 & t \in [0, T] \\
  u(0) &= u_0
\end{aligned}
\]

where

- \( A : D(A) \subset X \rightarrow X \) is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators on \( X \)
- \( B : D(B) \subset X \rightarrow X \) is a linear operator on \( X \) that can be either bounded (locally distributed control) or unbounded (boundary control)
- \( p : [0, T] \rightarrow X \) is the control
Bilinear control system

\[
\begin{cases}
  u'(t) + Au(t) + p(t)Bu(t) = 0 & t \in [0, T] \\
  u(0) = u_0
\end{cases}
\]

In this talk:
- the state space \((X, \langle \cdot, \cdot \rangle)\) is a Hilbert space
- \(B : X \to X\) is a bounded linear operator
- \(p : [0, T] \to \mathbb{R}\) is scalar function
What are the difficulties?

The map $\Phi : p \mapsto u$ is

**Additive** control:

$$\begin{cases} 
  u' + Au + Bp = 0 \\
  u(0) = u_0 
\end{cases}$$

**Bilinear** control:

$$\begin{cases} 
  u' + Au + pBu = 0 \\
  u(0) = u_0 
\end{cases}$$
What are the difficulties?

The map $\Phi : \mathbf{p} \mapsto u$ is

**Additive** control:
\[
\begin{align*}
  u' + Au + B\mathbf{p} &= 0 \\
  u(0) &= u_0
\end{align*}
\]

**Bilinear** control:
\[
\begin{align*}
  u' + Au + \mathbf{p}Bu &= 0 \\
  u(0) &= u_0
\end{align*}
\]

↓

**linear**

↓

**nonlinear**
An obstruction to exact controllability

**Bilinear control:**

\[
\begin{aligned}
&u' + Au + pBu = 0 \\
&u(0) = u_0
\end{aligned}
\]  

(1)

Let \( u_0 \in X \) and denote by \( u(\cdot; p, u_0) \) the unique solution of (1) for \( p \in L^1_{loc}(0, \infty) \).

**Theorem (Ball, Marsden, Slemrod 1982)**

*If \( \dim X = \infty \), then the attainable set from \( u_0 \)*

\[
S(u_0) = \{ u(t; p, u_0); t \geq 0, p \in L^1_{loc}(0, \infty) \}
\]

*is contained in a countable union of compact subsets of \( X \). So, \( X \setminus S(u_0) \) is dense.*
Motivations

Bilinear controls enter the system equations as coefficients changing (at least some of) the principal parameters of the process at hand

Examples

- by embedded *smart* alloys, the natural frequency response of a beam can be changed
- the rate of a chemical reaction can be altered by various catalysts and/or by the speed at which the reaction ingredients are mechanically mixed
A simplified model of a nuclear chain reaction

A chain reaction refers to a process in which neutrons released in fission produce an additional fission in at least one further nucleus. This nucleus in turn produces neutrons, and the process repeats. The process may be controlled (nuclear power) or uncontrolled (nuclear weapons).

\[
 u_t = a^2 \Delta u + v(t, x)u
\]

- \(u(t, x) \geq 0\) neutron density in the reaction
- \(v(t, x) > 0\) neutron amount in the surrounding medium
- \(v(t, x)u\) neutrons provided by the collision of the particles in the reaction with the surrounding medium
Schrödinger equation

The Schrödinger equation is a linear partial differential equation that describes the wave function or state function of a quantum-mechanical system

\[ i\psi_t = -\Delta \psi - p(t)\mu(x)\psi \]

- \(\psi\) wave function of a particle
- \(p\) amplitude of the electric field
- \(\mu\) dipolar moment of the particle
References

- J.M. Ball, J.E. Marsden, and M. Slemrod. Controllability for distributed bilinear systems. SIAM Journal on Control and Optimization


- K. Beauchard. Local controllability and non-controllability for a 1d wave equation with bilinear control. Journal of Differential Equations

- A.Y. Khapalov. Global non-negative controllability of the semilinear parabolic equation governed by bilinear control. ESAIM: Control, Optimisation and Calculus of Variations,

References

• J.M. Ball, J.E. Marsden, and M. Slemrod. Controllability for distributed bilinear systems. SIAM Journal on Control and Optimization

• attainability
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- J.M. Ball, J.E. Marsden, and M. Slemrod. Controllability for distributed bilinear systems. SIAM Journal on Control and Optimization
- **attainability**
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- J.M. Ball, J.E. Marsden, and M. Slemrod. Controllability for distributed bilinear systems. SIAM Journal on Control and Optimization
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• **approximate controllability**
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A look at stabilization
A look at stabilization

Figure: Jean-Pierre inspired by James Stewart
Notions of stabilizability

\[
\begin{cases}
    u' + Au + pBu = 0 \quad (t > 0) \\
    u(0) = u_0
\end{cases}
\] (2)
Notions of stabilizability

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\begin{aligned}
\begin{cases}
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\end{cases}
\end{aligned}
\] (2)

Let \( \bar{p} \in L^1_{loc}(0, \infty) \) and let \( \bar{u}_0 \in X \)

Definitions

• (2) is \textit{locally stabilizable to} \( u(\cdot; \bar{u}_0, \bar{p}) \) if \( \exists \delta > 0 \) such that for all \( u_0 \in B_\delta(\bar{u}_0) \) there exists \( p \in L^1_{loc}(0, \infty) \) such that

\[
\lim_{t \to +\infty} ||u(t; u_0, p) - u(t; \bar{u}_0, \bar{p})|| = 0
\]
Notions of stabilizability

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\begin{cases}
  u' + Au + pBu = 0 & (t > 0) \\
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**Definitions**

- (2) is **locally stabilizable to** \( u(\cdot; \bar{u}_0, \bar{p}) \) if \( \exists \delta > 0 \) such that for all \( u_0 \in B_{\delta}(\bar{u}_0) \) there exists \( p \in L^1_{\text{loc}}(0, \infty) \) such that

  \[
  \lim_{t \to +\infty} ||u(t; u_0, p) - u(t; \bar{u}_0, \bar{p})|| = 0
  \]

- (2) is **locally exponentially stabilizable to** \( u(\cdot; \bar{u}_0, \bar{p}) \) if \( \exists M, \delta, \rho > 0 \) such that for all \( u_0 \in B_{\delta}(\bar{u}_0) \) there exist \( p \in L^1_{\text{loc}}(0, \infty) \) satisfying

  \[
  ||u(t; u_0, p) - u(t; \bar{u}_0, \bar{p})|| \leq Me^{-\rho t} \quad \forall t > 0
  \]
Notions of stabilizability

\[
\begin{cases}
    u' + Au + pBu = 0 \quad (t > 0) \\
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**Definitions**

- (2) is **locally stabilizable to** \( u(\cdot; \bar{u}_0, \bar{p}) \) if \( \exists \delta > 0 \) such that for all \( u_0 \in B_\delta(\bar{u}_0) \) there exists \( p \in L^1_{loc}(0, \infty) \) such that
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  \[
  ||u(t; u_0, p) - u(t; \bar{u}_0, \bar{p})|| \leq Me^{-\rho t} \quad \forall t > 0
  \]
- (2) is **locally superexponentially stabilizable to** \( u(\cdot; \bar{u}_0, \bar{p}) \) if for any \( \rho > 0 \) there exists \( \delta > 0 \) such that, \( \forall u_0 \in B_\delta(\bar{u}_0) \), it holds that
  \[
  ||u(t; u_0, p) - \bar{u}(t; \bar{u}_0, \bar{p})|| \leq Me^{-\rho e^{\omega t}} \quad \forall t > 0
  \]

for some constants \( \exists M, \omega > 0 \)
Assumptions

Let \((X, \langle \cdot, \cdot \rangle)\) be a separable Hilbert space and \(A : D(A) \subset X \rightarrow X\) a densely defined linear operator with the following properties:

(a) \(A\) is self-adjoint

(b) \(\langle Ax, x \rangle \geq 0\) \(\forall x \in D(A)\) \hspace{1cm} (3)

(c) \(D(A) \subset X\) is compact
Assumptions

Let \((X, \langle \cdot, \cdot \rangle)\) be a separable Hilbert space and \(A : D(A) \subset X \to X\) a densely defined linear operator with the following properties:

\begin{enumerate}[(a)]
  
  \begin{align}
    (a) & \quad A \text{ is self-adjoint} \\
    (b) & \quad \langle Ax, x \rangle \geq 0 \quad \forall x \in D(A) \\
    (c) & \quad D(A) \subseteq X \text{ is compact}
  \end{align}
\end{enumerate}

\[\downarrow\]

1. \(X\) has an orthonormal basis \(\{\varphi_k\}_{k \in \mathbb{N}^*}\) of eigenvectors of \(A\)
2. the eigenvalues \(\{\lambda_k\}_{k \in \mathbb{N}^*}\) of \(A\) are nonnegative and \(\lambda_k \to +\infty\) as \(k \to +\infty\)
3. \(-A\) generates a strongly continuous semigroup of contractions \(e^{-tA}\)
Preliminaries

Given $T > 0$, consider the bilinear control problem

$$
\begin{align*}
    u'(t) + Au(t) + p(t)Bu(t) &= 0, \quad t \in [0, T] \\
    u(0) &= u_0
\end{align*}
$$

where $B \in \mathcal{L}(X)$ and $p \in L^2(0, T)$

Consider system (4) with $p = 0$:

$$
\begin{align*}
    u'(t) + Au(t) &= 0, \quad t \in [0, T] \\
    u(0) &= \phi_1
\end{align*}
$$

The solution $\psi_1(t) = e^{-\lambda_1 t}\phi_1\phi_1\phi_1$ is called the ground state solution

Remark

If $\langle A Ax, x \rangle \geq \nu |x|^2$ ($\nu > 0$), then $p = 0$ yields $\|u(t) - \psi_1(t)\| \leq e^{-\nu t}\|u_0 - \phi_1\phi_1\phi_1\|

So, (4) is locally exponentially (but not superexponentially) stabilizable to $\psi_1$.
Preliminaries

Given $T > 0$, consider the bilinear control problem

$$\begin{align*}
    \begin{cases}
    u'(t) + Au(t) + p(t)Bu(t) = 0, & t \in [0, T] \\
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    \end{cases}
\end{align*}$$

(4)

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\end{align*}$$

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Remark

If $\langle A x, x \rangle \geq \nu |x|^2$ ($\nu > 0$), then $p = 0$ yields

\[
\|u(t) - \psi_1(t)\| = \|e^{-tA} u_0 - e^{-tA} \varphi_1\| \leq e^{-\nu t} \|u_0 - \varphi_1\|
\]

So, (4) is locally exponentially (but not superexponentially) stabilizable to $\psi_1$. 
Superexponential stabilizability

\[
\begin{aligned}
&u'(t) + Au(t) + p(t)Bu(t) = 0 \quad (t > 0) \\
&u(0) = u_0
\end{aligned}
\]
Superexponential stabilizability

\[
\begin{cases}
    u'(t) + Au(t) + p(t)Bu(t) = 0 \quad (t > 0) \\
    u(0) = u_0
\end{cases}
\]

Theorem

Suppose there exists a constant \( \gamma > 0 \) such that the eigenvalues of \( A \) fulfill the gap condition

\[
\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} \geq \gamma \quad \forall k \geq 1
\]

where \( \lambda_k \) are the eigenvalues of \( A \).
Superexponential stabilizability

\[
\begin{cases}
  u'(t) + Au(t) + p(t)Bu(t) = 0 & (t > 0) \\
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Theorem

Suppose there exists a constant \( \gamma > 0 \) such that the eigenvalues of \( A \) fulfill the gap condition

\[
\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} \geq \gamma \quad \forall k \geq 1
\]

Let \( B : X \to X \) be a linear bounded operator with the following properties:

\[
\langle B\varphi_1, \varphi_k \rangle \neq 0 \quad \forall k \geq 1 \quad \& \quad \exists \tau > 0 \quad \text{such that} \quad \sum_{k=1}^{\infty} e^{-2\lambda_k \tau} \frac{1}{|\langle B\varphi_1, \varphi_k \rangle|^2} < \infty \quad (\star)
\]
Superexponential stabilizability

\[
\begin{aligned}
\left\{
\begin{array}{l}
    u'(t) + A u(t) + p(t) B u(t) = 0 \quad (t > 0) \\
    u(0) = u_0
\end{array}
\right.
\end{aligned}
\]

**Theorem**

Suppose there exists a constant \( \gamma > 0 \) such that the eigenvalues of \( A \) fulfill the gap condition

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\langle B \varphi_1, \varphi_k \rangle \neq 0 \quad \forall k \geq 1 \quad \& \quad \exists \tau > 0 \text{ such that } \sum_{k=1}^{\infty} \frac{e^{-2\lambda_k \tau}}{|\langle B \varphi_1, \varphi_k \rangle|^2} < \infty \quad (\star)
\]

Then, \( \forall \rho > 0, \exists R > 0 \) such that any \( u_0 \in B_R(\varphi_1) \) admits a control \( p \in L^2_{loc}(0, \infty) \) such that

\[
||u(t) - \psi_1(t)|| \leq M e^{-\rho e^{\omega t} - \lambda_1 t} \quad \forall t \geq 0
\]

where \( M \) and \( \omega \) are positive constants depending only on \( A \) and \( B \)
Sketch of the proof, $\lambda_1 = 0$

Fix $T > \tau$ where $\tau > 0$ is given by $(\ast)$

\[
\begin{cases}
  u'(t) + Au(t) + p(t)Bu(t) = 0, & t \in [0, T] \\
  u(0) = u_0
\end{cases}
\begin{cases}
  \psi_1'(t) + A\psi_1(t) = 0, & t \in [0, T] \\
  \psi_1(0) = \varphi_1
\end{cases}
\]

$\bar{v}(t; v_0, p) = u(t; v_0, p)$
Sketch of the proof, $\lambda_1 = 0$

Fix $T > \tau$ where $\tau > 0$ is given by $(\star)$

\[
\begin{cases}
    u'(t) + Au(t) + p(t)Bu(t) = 0, & t \in [0, T] \\
    u(0) = u_0,
\end{cases}
\]

\[
\begin{cases}
    \psi'_1(t) + A\psi_1(t) = 0, & t \in [0, T] \\
    \psi_1(0) = \varphi_1.
\end{cases}
\]

\[v := u - \psi_1\]
Sketch of the proof, $\lambda_1 = 0$

\[
\begin{aligned}
\left\{ \begin{array}{l}
 v'(t) + Av(t) + p(t)Bv(t) + p(t)B\psi_1(t) = 0, \\
 v(0) = v_0 = u_0 - \varphi_1,
\end{array} \right.
\end{aligned}
\]

\[t \in [0, T] \quad \text{(where } T > \tau)\]
Sketch of the proof, $\lambda_1 = 0$

\[ \begin{aligned}
&\begin{cases}
\nu'(t) + A\nu(t) + p(t)B\nu(t) + p(t)B\psi_1(t) = 0, \\
\nu(0) = \nu_0 = u_0 - \varphi_1,
\end{cases}
\quad \begin{cases}
\bar{\nu}'(t) + A\bar{\nu}(t) + p(t)B\psi_1(t) = 0, \\
\bar{\nu}(0) = \nu_0.
\end{cases}
\end{aligned} \]

$t \in [0, T]$ (where $T > \tau$)
Sketch of the proof, $\lambda_1 = 0$

\[ \begin{align*}
\begin{cases}
    v'(t) + Av(t) + p(t)Bv(t) + p(t)B\psi_1(t) = 0, \\
v(0) = v_0 = u_0 - \varphi_1,
\end{cases}
\end{align*} \quad \begin{align*}
\begin{cases}
    \bar{v}'(t) + A\bar{v}(t) + p(t)B\psi_1(t) = 0, \\
\bar{v}(0) = v_0.
\end{cases}
\end{align*} \quad t \in [0, T] \quad (\text{where } T > \tau) \]

$\nu_0$
Sketch of the proof, $\lambda_1 = 0$

\[
\begin{aligned}
\begin{cases}
    v'(t) + Av(t) + p(t)Bv(t) + p(t)B\psi_1(t) = 0, \\
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\end{cases}
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\end{cases}
\end{aligned}
\]

$t \in [0, T]$ (where $T > \tau$)
Sketch of the proof, $\lambda_1 = 0$

\[
\begin{aligned}
\begin{cases}
    v'(t) + Av(t) + p(t)Bv(t) + p(t)B\psi_1(t) = 0, \\
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\end{aligned}
\quad\begin{aligned}
\begin{cases}
    \bar{v}'(t) + A\bar{v}(t) + p(t)B\psi_1(t) = 0, \\
\bar{v}(0) = v_0.
\end{cases}
\end{aligned}
\]

$t \in [0, T]$ (where $T > \tau$)

\[\|p\|_{L^2(0, T)} \leq \Lambda_T \|v_0\|\]
Sketch of the proof, $\lambda_1 = 0$

\[
\begin{align*}
\{ & \quad v'(t) + Av(t) + p(t)Bv(t) + p(t)B\psi_1(t) = 0, \\
& \quad \bar{v}'(t) + A\bar{v}(t) + p(t)B\psi_1(t) = 0, \\
& \quad v(0) = v_0 = u_0 - \varphi_1, \\
& \quad \bar{v}(0) = \bar{v}_0.
\end{align*}
\]

$t \in [0, T]$ (where $T > \tau$)

\[
|p|_{L^2(0, T)} \leq \Lambda_T |v_0|
\]
Sketch of the proof, $\lambda_1 = 0$

\[
\begin{align*}
\begin{cases}
    v'(t) + Av(t) + p(t)Bv(t) + p(t)B\psi_1(t) = 0, \\
    \nu(0) = \nu_0 = u_0 - \varphi_1,
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
    \bar{v}'(t) + A\bar{v}(t) + p(t)B\psi_1(t) = 0, \\
    \bar{v}(0) = \nu_0.
\end{cases}
\end{align*}
\]

$t \in [0, T]$ \hspace{1cm} (where $T > \tau$)

\[
\|p\|_{L^2(0, T)} \leq \Lambda_T \|\nu_0\| \quad \|v(T) - \bar{v}(T)\| = \|v(T)\| \leq K_T \|\nu_0\|^2.
\]
Sketch of the proof, $\lambda_1 = 0$

\[
\begin{align*}
\begin{cases} 
    v'(t) + Av(t) + p(t)Bv(t) + p(t)B\psi_1(t) = 0, \\
    v(T) = v_T,
\end{cases} & \quad \begin{cases} 
    \bar{v}'(t) + A\bar{v}(t) + p(t)B\psi_1(t) = 0, \\
    \bar{v}(T) = v_T.
\end{cases}
\end{align*}
\]

$t \in [T, 2T]$ (where $T > \tau$)

$v_0$

$v(T; v_0, p)$

$\bar{v}(t; v_0, p)$

$0$
Sketch of the proof, $\lambda_1 = 0$

\[
\begin{align*}
\begin{cases}
  v'(t) + A v(t) + p(t) B v(t) + p(t) B \psi_1(t) = 0, \\
  v(T) = v_T,
\end{cases}
\end{align*}
\]

\[
\begin{cases}
  \bar{v}'(t) + A \bar{v}(t) + p(t) B \psi_1(t) = 0, \\
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\end{cases}
\]

$t \in [T, 2T]$ (where $T > \tau$)
Sketch of the proof, $\lambda_1 = 0$

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\begin{cases}
  v'(t) + Av(t) + p(t)Bv(t) + p(t)B\psi_1(t) = 0, \\
  \bar{v}(T) = v_T,
\end{cases}
\]

\[
\begin{cases}
  \bar{v}(t)' + A\bar{v}(t) + p(t)B\psi_1(t) = 0, \\
  \bar{v}(T) = v_T.
\end{cases}
\]

$t \in [T, 2T]$ (where $T > \tau$)

\[\|p\|_{L^2(T, 2T)} \leq \Lambda_T \|v(T)\|\]
Sketch of the proof, $\lambda_1 = 0$

\[
\begin{cases}
    v'(t) + Av(t) + p(t)Bv(t) + p(t)B\psi_1(t) = 0, \\
    v(T) = v_T,
\end{cases}
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\end{cases}
\]

$t \in [T, 2T]$ (where $T > \tau$)

\[
\|p\|_{L^2(T, 2T)} \leq \Lambda_T \|v(T)\|
\]
Sketch of the proof, $\lambda_1 = 0$

\[
\begin{cases}
    \dot{v}(t) + A v(t) + p(t) B v(t) + p(t) B \psi_1(t) = 0, \\
    v(T) = v_T,
\end{cases}
\]

\[
\begin{cases}
    \bar{v}(t)' + A \bar{v}(t) + p(t) B \psi_1(t) = 0, \\
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\end{cases}
\]

$t \in [T, 2T]$
Sketch of the proof, $\lambda_1 = 0$

\[
\begin{align*}
\begin{cases}
    v'(t) + A v(t) + p(t) B v(t) + p(t) B \psi_1(t) = 0, \\
    v(2T) = v_{2T},
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
    \bar{v}'(t) + A \bar{v}(t) + p(t) B \psi_1(t) = 0, \\
    \bar{v}(2T) = v_{2T}.
\end{cases}
\end{align*}
\]

\[t \in [2T, 3T]\] (where $T > \tau$)
Sketch of the proof, $\lambda_1 = 0$

\[
\begin{aligned}
\begin{cases}
\dot{v}(t) + Av(t) + p(t)Bv(t) + p(t)B\psi_1(t) = 0, \\
v(2T) = v_{2T},
\end{cases}
&\quad\begin{cases}
\dot{\bar{v}}(t) + A\bar{v}(t) + p(t)B\psi_1(t) = 0, \\
\bar{v}(2T) = v_{2T}.
\end{cases}
\end{aligned}
\]

$t \in [2T, 3T]$ (where $T > \tau$)
Sketch of the proof, $\lambda_1 = 0$

$$\begin{cases} v'(t) + Av(t) + p(t)Bv(t) + p(t)B\psi_1(t) = 0, \\ v(2T) = v_{2T}, \end{cases} \quad \begin{cases} \bar{v}'(t) + A\bar{v}(t) + p(t)B\psi_1(t) = 0, \\ \bar{v}(2T) = v_{2T}. \end{cases}$$

$t \in [2T, 3T]$ (where $T > \tau$)

\[ \|p\|_{L^2(2T, 3T)} \leq \Lambda_T \|v(2T)\| \]
Sketch of the proof, $\lambda_1 = 0$

\[
\begin{cases}
    v'(t) + Av(t) + p(t)Bv(t) + p(t)B\psi_1(t) = 0, \\
    v(2T) = v_{2T},
\end{cases}
\]

\[
\begin{cases}
    \bar{v}'(t) + A\bar{v}(t) + p(t)B\psi_1(t) = 0, \\
    \bar{v}(2T) = v_{2T}.
\end{cases}
\]

$t \in [2T, 3T]$ (where $T > \tau$)

\[
||p||_{L^2(2T, 3T)} \leq \Lambda_T ||v(2T)||
\]
Sketch of the proof, $\lambda_1 = 0$

\[
\begin{align*}
   & \left\{ \begin{array}{l}
          v'(t) + Av(t) + p(t)Bv(t) + p(t)B\psi_1(t) = 0, \\
          v(2T) = v_{2T},
        \end{array} \right. \\
   & \left\{ \begin{array}{l}
          \bar{v}(t)' + A\bar{v}(t) + p(t)B\psi_1(t) = 0, \\
          \bar{v}(2T) = v_{2T}.
        \end{array} \right.
\end{align*}
\]

\[t \in [2T, 3T] \quad \text{(where } T > \tau)\]

\[\|p\|_{L^2(2T, 3T)} \leq \Lambda_T \|v(2T)\| \quad \|v - \bar{v}\|(3T) = \|v(3T)\| \leq K_T \|v(2T)\|^2 \leq (1/K_T)(K_T \|v_0\|)^2.\]
The moment method

\[
\begin{aligned}
\begin{cases}
\nu'(t) + A\nu(t) + p(t)B\nu(t) + p(t)B\psi_1(t) = 0 \\
\nu(0) = \nu_0 = u_0 - \varphi_1
\end{cases}
\end{aligned}
\]

Lemma

Let \( T > \tau \). Then there exists a control \( p \in L^2(0, T) \) such that \( \bar{\nu}(T) = 0 \). Moreover

\[
\|p\|_{L^2(0, T)} \leq \Lambda_T \|\nu_0\|
\]

\((P)\)
The moment method

\[
\begin{aligned}
\begin{cases}
  v'(t) + A v(t) + p(t) B v(t) + p(t) B \psi_1(t) = 0 \\
  v(0) = v_0 = u_0 - \phi_1
\end{cases}
\quad
\begin{cases}
  \bar{v}(t)' + A \bar{v}(t) + p(t) B \psi_1(t) = 0 \\
  \bar{v}(0) = v_0
\end{cases}
\end{aligned}
\]

Lemma

Let \( T > \tau \). Then there exists a control \( p \in L^2(0, T) \) such that \( \bar{v}(T) = 0 \). Moreover

\[
\|p\|_{L^2(0, T)} \leq \Lambda_T \|v_0\| \tag{P}
\]

Proof. By looking at the Fourier expansion of \( \bar{v} \), we conclude that \( \bar{v}(T) = 0 \) is equivalent to

\[
\int_0^T e^{\lambda_k s} p(s) ds = \frac{\langle v_0, \varphi_k \rangle}{\langle B \varphi_1, \varphi_k \rangle}
\]
The moment method

\[
\begin{aligned}
&v'(t) + A v(t) + p(t) B v(t) + p(t) B \psi_1(t) = 0 \\
&v(0) = v_0 = u_0 - \varphi_1 \\
&\bar{v}'(t) + A \bar{v}(t) + p(t) B \psi_1(t) = 0 \\
&\bar{v}(0) = v_0
\end{aligned}
\]

**Lemma**

Let \( T > \tau \). Then there exists a control \( p \in L^2(0, T) \) such that \( \bar{v}(T) = 0 \). Moreover

\[
\|p\|_{L^2(0, T)} \leq \Lambda_T \|v_0\| \tag{P}
\]

**Proof.** By looking at the Fourier expansion of \( \bar{v} \), we conclude that \( \bar{v}(T) = 0 \) is equivalent to

\[
\int_0^T e^{\lambda_k s} p(s) ds = \frac{\langle v_0, \varphi_k \rangle}{\langle B \varphi_1, \varphi_k \rangle} \quad \Rightarrow \quad p(s) = \sum_{k=1}^{\infty} \frac{\langle v_0, \varphi_k \rangle}{\langle B \varphi_1, \varphi_k \rangle} \sigma_k(s)
\]

where \( \{\sigma_k\}_{k \geq 1} \) is biorthogonal to \( \{e^{\lambda_k s}\}_{k \geq 1} \).
The moment method

{\begin{align*}
v'(t) + A v(t) + p(t) B v(t) + p(t) B \psi_1(t) &= 0 \\
v(0) &= v_0 = u_0 - \varphi_1
\end{align*}}

{\begin{align*}
\bar{v}'(t) + A \bar{v}(t) + p(t) B \psi_1(t) &= 0 \\
\bar{v}(0) &= v_0
\end{align*}}

Lemma

Let $T > \tau$. Then there exists a control $p \in L^2(0, T)$ such that $\bar{v}(T) = 0$. Moreover

$$\|p\|_{L^2(0, T)} \leq \Lambda_T \|v_0\|$$

\textbf{Proof}. By looking at the Fourier expansion of $\bar{v}$, we conclude that $\bar{v}(T) = 0$ is equivalent to

$$\int_0^T e^{\lambda_k s} p(s) ds = \frac{\langle v_0, \varphi_k \rangle}{\langle B \varphi_1, \varphi_k \rangle} \implies p(s) = \sum_{k=1}^{\infty} \frac{\langle v_0, \varphi_k \rangle}{\langle B \varphi_1, \varphi_k \rangle} \sigma_k(s)$$

where $\{\sigma_k\}_{k \geq 1}$ is biorthogonal to $\{e^{\lambda_k s}\}_{k \geq 1}$. Bounds for $\{\sigma_k\}_{k \geq 1}$ and $(\star)$ can be used to obtain $(P)$, thus ensuring that the above series converges in $L^2(0, T)$.
Sketch of the proof, $\lambda_1 = 0$

- \[ \|v(nT)\| \leq \frac{1}{K_T}(K_T\|v_0\|)^{2^n} \quad \forall n \geq 0 \quad (8) \]

- In any time interval $[nT, (n + 1)T]$ we prove that
\[ \|v(t)\| \leq C_T\|v(nT)\| \quad nT \leq t \leq (n + 1)T \quad (9) \]
Sketch of the proof, $\lambda_1 = 0$

- $\|v(nT)\| \leq \frac{1}{K_T} (K_T \|v_0\|)^{2^n} \quad \forall n \geq 0 \tag{8}$

- In any time interval $[nT, (n + 1)T]$ we prove that
  $$\|v(t)\| \leq C_T \|v(nT)\| \quad nT \leq t \leq (n + 1)T \tag{9}$$

- Let $\theta \in (0,1)$ and $\|v_0\| \leq \frac{\theta}{K_T}$. Combining (8) and (9), we obtain
  $$\|u(t) - \psi_1(t)\| \leq \frac{C_T}{K_T} \theta^{2t/T - 1} \quad \forall t \geq 0$$
Sketch of the proof, $\lambda_1 = 0$

- $\|v(nT)\| \leq \frac{1}{K_T} (K_T \|v_0\|)^{2n} \quad \forall n \geq 0$ \hspace{1cm} (8)

- In any time interval $[nT, (n + 1)T]$ we prove that
  \[ \|v(t)\| \leq C_T \|v(nT)\| \quad nT \leq t \leq (n + 1)T \] \hspace{1cm} (9)

- Let $\theta \in (0, 1)$ and $\|v_0\| \leq \frac{\theta}{K_T}$. Combining (8) and (9), we obtain
  \[ \|u(t) - \varphi_1(t)\| \leq \frac{C_T}{K_T} \theta^{2T/\theta^{T-1}} \quad \forall t \geq 0 \]

- Let $\theta \in (0, 1)$ and let $\rho > 0$ be the value for which $\theta = e^{-2\rho}$. Then $\exists R_\rho > 0$ such that, for all $\|u_0 - \varphi_1\| \leq R_\rho$, we have
  \[ \|u(t) - \varphi_1\| \leq M_T e^{-\rho \omega_T t} \quad \forall t \geq 0 \]
  with $M_T, \omega_T > 0$ suitable constants
Sketch of the proof, $\lambda_1 > 0$

We introduce the operator

$$A_1 := A - \lambda_1 I.$$ 

Notice that $A_1 : D(A_1) \subset X \to X$ is self-adjoint, accretive and $-A_1$ generates a strongly continuous semigroup of contractions. Its eigenvalues are given by

$$\mu_k = \lambda_k - \lambda_1, \quad \forall k \in \mathbb{N}^*$$

(in particular, $\mu_1 = 0$) and $A_1$ has the same eigenfunctions as $A$, $\{\varphi_k\}_{k \geq 1}$. Moreover, the family $\{\mu_k\}_{k \geq 1}$ satisfies the same gap condition that is satisfied by the eigenvalues of $A$.
Sketch of the proof, $\lambda_1 > 0$

We define $z(t) = e^{\lambda_1 t}u(t)$. Then

$$\begin{cases} z'(t) + A_1 z(t) + p(t)B z(t) = 0 & (t > 0) \\
z(0) = u_0. \end{cases}$$

So, we can apply the previous analysis to this problem:

$$||z(t) - \varphi_1|| \leq M_T e^{-\rho \omega T t} \quad \forall t \geq 0$$

Therefore, returning to $u$,

$$||u(t) - \psi_1(t)|| = ||e^{-\lambda_1 t}z(t) - e^{-\lambda_1 t}\varphi_1|| = e^{-\lambda_1 t}||z(t) - \varphi_1|| \leq M_T e^{-(\rho \omega T t + \lambda_1 t)} \quad \forall t \geq 0$$
Example 1 (heat eqn, Dirichlet bc)

Consider the bilinear control system

\[
\begin{cases}
  u_t(t, x) - u_{xx}(t, x) + p(t) \mu(x) u(t, x) = 0, \\
  u(t, 0) = u(t, 1) = 0, \\
  u(0, x) = u_0(x)
\end{cases}
\]
Example 1 (heat eqn, Dirichlet bc)

Consider the bilinear control system

\[
\begin{align*}
    &u_t(t, x) - u_{xx}(t, x) + p(t)\mu(x)u(t, x) = 0, \quad (t, x) \in (0, \infty) \times (0, 1) \\
    &u(t, 0) = u(t, 1) = 0 \\
    &u(0, x) = u_0(x)
\end{align*}
\]
Example 1

Let $X = L^2(0,1)$ and consider the bilinear control system

$$\begin{cases} u_t(t) + Au(t) + p(t)Bu(t) = 0, & t \in (0, \infty) \\ u(0) = u_0(x) \end{cases} \quad (10)$$

where $A$ and $B$ are defined by

$$D(A) = H^2 \cap H^1_0(0,1), \quad A\varphi = -\frac{d^2\varphi}{dx^2}$$

$$B \in \mathcal{L}(X), \quad B\varphi = \mu\varphi \quad (11)$$
Example 1

Let $X = L^2(0, 1)$ and consider the bilinear control system

$$\begin{cases} u_t(t) + Au(t) + p(t)Bu(t) = 0, & t \in (0, \infty) \\ u(0) = u_0(x) \end{cases}$$

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(11)

- $A$ is a self-adjoint accretive operator and $-A$ generates an analytic $C^0$-semigroup.
- Eigenvalues and eigenvectors of $A$:

$$\lambda_k = (k\pi)^2, \quad \varphi_k(x) = \sqrt{2} \sin(k\pi x) \quad \forall k \geq 1$$
Example 1

Let $X = L^2(0, 1)$ and consider the bilinear control system

$$
\begin{align*}
& \begin{cases}
  u_t(t) + Au(t) + p(t) Bu(t) = 0, & t \in (0, \infty) \\
  u(0) = u_0(x)
\end{cases} \\
& \quad \quad \quad (10)
\end{align*}
$$

where $A$ and $B$ are defined by

$$
\begin{align*}
  & D(A) = H^2 \cap H^1_0(0, 1), \quad A\varphi = -\frac{d^2\varphi}{dx^2} \\
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$$

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\lambda_k = (k\pi)^2, \quad \varphi_k(x) = \sqrt{2} \sin(k\pi x) \quad \forall k \geq 1
$$

We want to study the superexponential stabilizability of (27)-(27) to the ground state solution

$$
\psi_1 = e^{-\lambda_1 t} \varphi_1
$$
Example 1

- gap condition:

\[
\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} = (k+1)\pi - k\pi = \pi \quad \forall k \geq 1
\]
Example 1

- gap condition:
  \[ \sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} = (k + 1)\pi - k\pi = \pi \quad \forall k \geq 1 \]

- estimate of the Fourier coefficients:
  \[
  \langle B\varphi_1, \varphi_k \rangle = \int_0^1 2\mu(x)\sin(\pi x)\sin(k\pi x)\,dx \\
  = \frac{4}{k^3} \left( (-1)^{k+1}\mu'(1) - \mu'(0) \right) + \\
  - \frac{\sqrt{2}}{(k\pi)^3} \int_0^1 (\mu(x)\varphi_1(x))''' \cos(k\pi x)\,dx
  \]
Example 1

- gap condition:
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  - \frac{\sqrt{2}}{(k\pi)^3} \int_0^1 (\mu(x)\varphi_1(x))''' \cos(k\pi x) dx
  \]

If \( \langle B\varphi_1, \varphi_k \rangle \neq 0 \ \forall k \in \mathbb{N}^* \) and \( \mu'(1) \pm \mu'(0) \neq 0 \), then we have

\[
|\langle B\varphi_1, \varphi_k \rangle| \geq C\lambda_k^{-3/2}, \quad \forall k \in \mathbb{N}^*.
\]
Example 1

- gap condition:
  \[ \sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} = (k + 1)\pi - k\pi = \pi \quad \forall k \geq 1 \]

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  \[ \langle B\varphi_1, \varphi_k \rangle = \int_0^1 2\mu(x) \sin(\pi x) \sin(k\pi x) dx \]
  \[ = \frac{4}{k^3} \left( (-1)^{k+1} \mu'(1) - \mu'(0) \right) + \]
  \[ - \frac{\sqrt{2}}{(k\pi)^3} \int_0^1 (\mu(x)\varphi_1(x))''' \cos(k\pi x) dx \]

If \( \langle B\varphi_1, \varphi_k \rangle \neq 0 \ \forall k \in \mathbb{N}^* \) and \( \mu'(1) \pm \mu'(0) \neq 0 \), then we have

\[ |\langle B\varphi_1, \varphi_k \rangle| \geq C\lambda_k^{-3/2}, \quad \forall k \in \mathbb{N}^*. \]

**EXAMPLE:** \[ B\varphi(x) = x^2\varphi(x) \]
Example 1: conclusion

- The series
  \[
  \sum_{k \in \mathbb{N}^*} \frac{e^{-2\lambda_k \tau}}{|\langle B\varphi_1, \varphi_k \rangle|^2}
  \]
  converges for all \( \tau > 0 \).
Example 1: conclusion

- The series

\[ \sum_{k \in \mathbb{N}^*} \frac{e^{-2\lambda_k \tau}}{|\langle B\varphi_1, \varphi_k \rangle|^2} \]

converges for all \( \tau > 0 \).

Therefore, our abstract result guarantees that

\[
\begin{cases}
  u_t(t, x) - u_{xx}(t, x) + p(t)x^2 u(t, x) = 0 & (t, x) \in (0, \infty) \times (0, 1) \\
  u(t, 0) = u(t, 1) = 0 \\
  u(0, x) = u_0(x)
\end{cases}
\]

is locally superexponentially stabilizable to \( \psi_1(t) = e^{-\lambda_1 t}\varphi_1 \)
Example 2 (heat eqn, Neumann bc)

Let $\Omega = [0, 1]$ and consider the bilinear control system

\[
\begin{align*}
    u_t(t, x) & - u_{xx}(t, x) + p(t) \mu(x) u(t, x) = 0, \\
    u_x(t, 0) &= u_x(t, 1) = 0, \\
    u(0, x) &= u_0(x)
\end{align*}
\]
Example 2 (heat eqn, Neumann bc)

Let $\Omega = [0, 1]$ and consider the bilinear control system

$$\begin{align*}
&u_t(t, x) - u_{xx}(t, x) + p(t)\mu(x)u(t, x) = 0, \quad (t, x) \in (0, \infty) \times (0, 1) \\
&u_x(t, 0) = u_x(t, 1) = 0 \\
&u(0, x) = u_0(x)
\end{align*}$$
Example 2

Let $X = L^2(0, 1)$ and consider the bilinear control system

$$\begin{cases} 
  u_t(t) + Au(t) + p(t)Bu(t) = 0, & t \in (0, \infty) \\
  u(0) = u_0(x).
\end{cases}$$

where $A$ and $B$ are defined by

$$D(A) = \left\{ \varphi \in H^2(0, 1) : \frac{d\varphi}{dx}(0) = 0 = \frac{d\varphi}{dx}(1) \right\}, \quad A\varphi = -\frac{d^2\varphi}{dx^2}$$

$$B \in \mathcal{L}(X), \quad B\varphi = \mu \varphi$$
Example 2

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  u(0) = u_0(x).
\end{cases}
\end{aligned}
$$

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$$

$B \in \mathcal{L}(X), \quad B\varphi = \mu \varphi$

- $A$ is a **self-adjoint accretive** operator and $-A$ generates an **analytic** $C^0$-semigroup.
- Eigenvalues and eigenvectors of $A$:

  $$
  \lambda_0 = 0, \quad \varphi_0 = 1
  $$
  $$
  \lambda_k = (k\pi)^2, \quad \varphi_k(x) = \sqrt{2} \cos(k\pi x), \quad \forall k \geq 1
  $$
Example 2

- gap condition:

\[ \sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} = (k + 1)\pi - k\pi = \pi \quad \forall k \geq 1 \]
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If \( \langle B\varphi_1, \varphi_k \rangle \neq 0 \ \forall k \in \mathbb{N}^* \) and \( \mu'(1) \pm \mu'(0) \neq 0 \), then we have

\[ |\langle B\varphi_1, \varphi_k \rangle| \geq C\lambda_k^{-1}, \quad \forall k \geq 1 \]
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EXAMPLE: \( B\varphi(x) = x^2 \varphi(x) \)
Example 2: conclusion

- The series

\[ \sum_{k \in \mathbb{N}^*} \frac{e^{-2\lambda_k \tau}}{|\langle B\varphi_1, \varphi_k \rangle|^2} \]

converges for all \( \tau > 0 \)
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Therefore, our abstract result guarantees that

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  u_t(t, x) - u_{xx}(t, x) + p(t)x^2u(t, x) = 0, & (t, x) \in (0, \infty) \times (0, 1) \\
  u_x(t, 0) = u_x(t, 1) \\
  u(0, x) = u_0(x)
\end{cases}
\]

is locally superexponentially stabilizable to \( \psi_0(t) = \varphi_0 \)
Example 3 (degenerate heat eqn)

\[
\begin{aligned}
&\text{Let } 0 \leq \alpha < 2 \text{ and consider the bilinear control system}

&\begin{aligned}
&u_{tt} - (x^\alpha u_x)_x + p(t)\mu(x)u = 0, \\
&(t, x) \in (0, \infty) \times (0, 1),
\end{aligned}

&u(t, 1) = 0, \\
&\begin{cases}
&u(t, 0) = 0, &\text{if } \alpha \in [0, 1), \\
&(x^\alpha u_x)(t, 0) = 0, &\text{if } \alpha \in [1, 2),
\end{cases}

&u(0, x) = u_0(x),
\end{aligned}
\]
Example 3 (degenerate heat eqn)

Let $0 \leq \alpha < 2$ and consider the bilinear control system

\[
\begin{align*}
  u_t - (x^\alpha u_x)_x + p(t)\mu(x)u &= 0, & (t, x) \in (0, \infty) \times (0, 1) \\
  u(t, 1) &= 0, & \\
  u(t, 0) &= 0, & \text{if } \alpha \in [0, 1), \\
  (x^\alpha u_x)(t, 0) &= 0, & \text{if } \alpha \in [1, 2) \\
  u(0, x) &= u_0(x)
\end{align*}
\]
Example 3: weakly degenerate case ($\alpha < 1$)

Let $X = L^2(0, 1)$ and consider the bilinear control system

\[
\begin{cases}
  u_t(t) + Au(t) + p(t)Bu(t) = 0, & t \in (0, \infty) \\
  u(0) = u_0(x).
\end{cases}
\]

where $A$ and $B$ are defined by

\[
D(A) = \left\{ u \in H^{1, 0}_{\alpha, 0}(0, 1) : x^{\alpha/2} \frac{du}{dx} \in H^1(0, 1) \right\}, \quad A\varphi = -\frac{d}{dx}(x^{\alpha} \frac{du}{dx})
\]

\[
B \in \mathcal{L}(X), \quad B\varphi = \mu \varphi
\]

where

\[
H^{1, 0}_{\alpha, 0}(0, 1) = \left\{ u \in L^2(0, 1) : u \in AC([0, 1]), x^{\alpha/2} \frac{du}{dx} \in L^2(0, 1), u(0) = 0, u(1) = 0 \right\}
\]
Example 3: weakly degenerate case ($\alpha < 1$)

Let $X = L^2(0, 1)$ and consider the bilinear control system

$$
\begin{cases}
  u_t(t) + Au(t) + p(t)Bu(t) = 0, & t \in (0, \infty) \\
  u(0) = u_0(x).
\end{cases}
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where $A$ and $B$ are defined by

$$
D(A) = \left\{ u \in H^1_{\alpha,0}(0, 1) : x^{\alpha} \frac{du}{dx} \in H^1(0, 1) \right\},
A\varphi = -\frac{d}{dx}(x^{\alpha} \frac{du}{dx})
$$

$$
B \in L(X),
B\varphi = \mu \varphi
$$

where

$$
H^1_{\alpha,0}(0, 1) = \left\{ u \in L^2(0, 1) : u \in AC([0, 1]), x^{\alpha/2} \frac{du}{dx} \in L^2(0, 1), u(0) = 0, u(1) = 0 \right\}
$$

Then

- $A$ is a self-adjoint accretive operator and $-A$ generates a $C^0$-semigroup of contractions.
Example 3: gap condition ($\alpha < 1$)

For any $\nu \geq 0$, we denote by

- $J_\nu$ the Bessel function of the first kind and order $\nu$
- $j_{\nu,1} < j_{\nu,2} < \cdots < j_{\nu,k} < \cdots$ the sequence of all positive zeros of $J_\nu$
Example 3: gap condition ($\alpha < 1$)

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Set

$$
\nu_\alpha := \frac{1 - \alpha}{2 - \alpha}, \quad k_\alpha := \frac{2 - \alpha}{2}.
$$
Example 3: gap condition \((\alpha < 1)\)

For any \(\nu \geq 0\), we denote by

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Set

\[
\nu_\alpha := \frac{1 - \alpha}{2 - \alpha}, \quad k_\alpha := \frac{2 - \alpha}{2}.
\]

Then eigenvalues and eigenvectors of \(A\) are given by:

\[
\lambda_{\alpha,k} = k_\alpha^2 j_{\alpha,k}^2 \quad (k \geq 1)
\]

\[
\varphi_{\alpha,k}(x) = \frac{\sqrt{2k_\alpha}}{|J'_{\nu_\alpha}(j_{\nu_\alpha,k})|} x^{(1-\alpha)/2} J_{\nu_\alpha} \left( j_{\nu_\alpha,k} x^{k_\alpha} \right) \quad (k \geq 1)
\]
Example 3: gap condition \((\alpha < 1)\)

For any \(\nu \geq 0\), we denote by

- \(J_\nu\) the Bessel function of the first kind and order \(\nu\)
- \(j_{\nu,1} < j_{\nu,2} < \cdots < j_{\nu,k} < \cdots\) the sequence of all positive zeros of \(J_\nu\)

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\varphi_{\alpha,k}(x) = \frac{\sqrt{2k_\alpha}}{|J'_\nu_{\alpha}(j_{\nu_\alpha,k})|} x^{(1-\alpha)/2} J_{\nu_\alpha} \left(j_{\nu_\alpha,k} x^{k_\alpha}\right) \quad (k \geq 1)
\]

So, the gap condition is satisfied:

\[
\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} = k_\alpha (j_{\nu_\alpha,k+1} - j_{\nu_\alpha,k}) \geq k_\alpha (j_{\nu_\alpha,2} - j_{\nu_\alpha,1}) \geq \frac{7}{16} \pi
\]
Example 3: conclusion

- Taking $\mu(x) = x^{2-\alpha}$ a long computation shows that

$$|\langle \mu \varphi_1, \varphi_k \rangle| \geq \frac{C}{\lambda_k^{3/2}} \quad \forall k \geq 1$$

for some $C > 0$

- So, the series

$$\sum_{k=1}^{\infty} \frac{e^{-2\lambda_k \tau}}{|\langle B \varphi_1, \varphi_k \rangle|^2}$$

converges for all $\tau > 0$
Example 3: conclusion

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- So, the series

$$\sum_{k=1}^{\infty} \frac{e^{-2\lambda_k \tau}}{|\langle B \varphi_1, \varphi_k \rangle|^2}$$

converges for all $\tau > 0$

Therefore, our abstract result guarantees that

$$\begin{cases}
  u_t(t, x) - u_{xx}(t, x) + p(t)x^{2-\alpha}u(t, x) = 0, \quad (t, x) \in (0, \infty) \times \Omega \\
  u_x(t, 0) = u_x(t, 1) \\
  u(0, x) = u_0(x)
\end{cases}$$

is locally superexponentially stabilizable to $\psi_1$
Theorem

Suppose there exists a constant \( \gamma > 0 \) such that the eigenvalues of \( A \) fulfill the gap condition

\[
\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} \geq \gamma \quad \forall k \geq 1
\]

Let \( B : X \to X \) be a linear bounded operator satisfying the following

\[
\langle B \varphi_1, \varphi_1 \rangle \neq 0 \quad \text{and} \quad \exists b, q > 0 \quad \text{such that} \quad |\langle B \varphi_1, \varphi_k \rangle| \geq b \quad \forall k > 1 \quad (\dagger)
\]

Then for any \( T > 0 \) there exists \( R_T > 0 \) such that, for any \( u_0 \in B_{R_T}(\varphi_1) \), the solution to \((S)\) can be steered to the ground state solution in time \( T \) by some control \( p \in L^2(0, T) \).

Notice that \((\dagger)\) is satisfied by all the above examples. Moreover \((\dagger) \Rightarrow \infty \sum_{k=1}^{\infty} e^{-2\lambda_k \tau} |\langle B \varphi_1, \varphi_k \rangle|^2 < \infty \quad \forall \tau > 0\)
Exact controllability to the ground state solution

\[
\begin{cases}
  u'(t) + Au(t) + p(t)Bu(t) = 0 & (t > 0) \\
  u(0) = u_0
\end{cases}
\tag{S}
\]

Theorem

Suppose there exists a constant \( \gamma > 0 \) such that the eigenvalues of \( A \) fulfill the gap condition

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\[
\langle B\varphi_1, \varphi_1 \rangle \neq 0 \quad \& \quad \exists b, q > 0 \quad \text{such that} \quad \lambda_k^q |\langle B\varphi_1, \varphi_k \rangle| \geq b \quad \forall k > 1
\] (**)
Exact controllability to the ground state solution

\[
\begin{cases}
  u'(t) + Au(t) + p(t)Bu(t) = 0 \quad (t > 0) \\
  u(0) = u_0
\end{cases}
\]  
(S)

**Theorem**

*Suppose there exists a constant \( \gamma > 0 \) such that the eigenvalues of \( A \) fulfill the gap condition*

\[
\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} \geq \gamma \quad \forall k \geq 1
\]

*Let \( B : X \to X \) be a linear bounded operator satisfying the following*

\[
\langle B\varphi_1, \varphi_1 \rangle \neq 0 \quad \& \quad \exists b, q > 0 \quad \text{such that} \quad \lambda_k^q |\langle B\varphi_1, \varphi_k \rangle| \geq b \quad \forall k > 1
\]  
(\ast\ast)

*Then for any \( T > 0 \) there exists \( R_T > 0 \) such that, for any \( u_0 \in B_{R_T}(\varphi_1) \), the solution to (S) can be steered to the ground state solution in time \( T \) by some control \( p \in L^2(0, T) \).*
Exact controllability to the ground state solution

\( \begin{cases} u'(t) + Au(t) + p(t)Bu(t) = 0 & (t > 0) \\ u(0) = u_0 \end{cases} \) \quad (S)

**Theorem**

Suppose there exists a constant \( \gamma > 0 \) such that the eigenvalues of \( A \) fulfill the gap condition

\[ \sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} \geq \gamma \quad \forall k \geq 1 \]

Let \( B : X \to X \) be a linear bounded operator satisfying the following

\[ \langle B\varphi_1, \varphi_1 \rangle \neq 0 \quad \& \quad \exists b, q > 0 \text{ such that } \lambda_k^q|\langle B\varphi_1, \varphi_k \rangle| \geq b \quad \forall k > 1 \quad (**) \]

Then for any \( T > 0 \) there exists \( R_T > 0 \) such that, for any \( u_0 \in B_{R_T}(\varphi_1) \), the solution to (S) can be steered to the ground state solution in time \( T \) by some control \( p \in L^2(0, T) \)

Notice that (**) is satisfied by all the above examples. Moreover

\[ (**) \implies \sum_{k=1}^{\infty} \frac{e^{-2\lambda_k \tau}}{|\langle B\varphi_1, \varphi_k \rangle|^2} < \infty \quad \forall \tau > 0 \]
Global exact controllability on a strip

\[
\begin{aligned}
&u'(t) + Au(t) + p(t)Bu(t) = 0 \quad (t > 0) \\
&u(0) = u_0
\end{aligned}
\]  \tag{S}

Theorem

Suppose there exists a constant \( \gamma > 0 \) such that the eigenvalues of \( A \) fulfill the gap condition

\[
\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} \geq \gamma \quad \forall \ k \geq 1
\]

Let \( B : X \to X \) be a linear bounded operator satisfying the following

\[
\langle B\phi_1 \phi_1 \phi_1, \phi_1 \phi_1 \phi_1 \rangle 
eq 0 \quad \exists b, q > 0 \quad \text{satisfying} \quad \lambda_q \geq \frac{b}{\langle B\phi_1 \phi_1 \phi_1, \phi_k \phi_k \phi_k \rangle} \quad \forall k > 1 \quad (\star \star)
\]

Then there exists \( r_1 > 0 \) such that for all \( 0 < r < r_1 \) and all \( R > 0 \) there exists \( T_{r, R} > 0 \) such that for all \( u_0 \in X \) in the strip

\[
|\langle u_0, \phi_1 \phi_1 \phi_1 \rangle - 1| \leq r,
\]

\[
||u_0 - \langle u_0, \phi_1 \phi_1 \phi_1 \rangle \phi_1 \phi_1 \phi_1|| \leq R,
\]

the solution to \((S)\) can be steered to the ground state solution \( \psi_1(t) = e^{-\lambda_1 t} \phi_1 \phi_1 \phi_1 \) in time \( T_{r, R} \) by some control \( p \in L^2(0, T_{r, R}) \).
Global exact controllability on a strip

\[
\begin{align*}
\left\{ \begin{array}{l}
u'(t) + Au(t) + p(t)Bu(t) = 0 \quad (t > 0) \\
u(0) = u_0
\end{array} \right. \\
\end{align*}
\]

\( \text{(S)} \)

**Theorem**

Suppose there exists a constant \( \gamma > 0 \) such that the eigenvalues of \( A \) fulfill the gap condition

\[
\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} \geq \gamma \quad \forall k \geq 1
\]

Let \( B : X \to X \) be a linear bounded operator satisfying the following

\[
\langle B\varphi_1, \varphi_1 \rangle \neq 0 \quad \& \quad \exists b, q > 0 \quad \text{such that} \quad \lambda_k^q |\langle B\varphi_1, \varphi_k \rangle| \geq b \quad \forall k > 1
\]

\( \text{(**)} \)
Global exact controllability on a strip

\[
\begin{cases}
  u'(t) + Au(t) + p(t)B u(t) = 0 \quad (t > 0) \\
  u(0) = u_0
\end{cases}
\] (S)

**Theorem**

Suppose there exists a constant \( \gamma > 0 \) such that the eigenvalues of \( A \) fulfill the gap condition

\[
\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} \geq \gamma \quad \forall k \geq 1
\]

Let \( B : X \to X \) be a linear bounded operator satisfying the following

\[
\langle B \varphi_1, \varphi_1 \rangle \neq 0 \quad \& \quad \exists b, q > 0 \text{ such that } \lambda_k^q \langle B \varphi_1, \varphi_k \rangle \geq b \quad \forall k > 1 \quad (\star \star)
\]

Then there exists \( r_1 > 0 \) such that for all \( 0 < r < r_1 \) and all \( R > 0 \) there exists \( T_{r,R} > 0 \) such that for all \( u_0 \in X \) in the strip

\[
\begin{align*}
|\langle u_0, \varphi_1 \rangle - 1| & \leq r, \\
||u_0 - \langle u_0, \varphi_1 \rangle \varphi_1|| & \leq R,
\end{align*}
\]

the solution to (S) can be steered to the ground state solution \( \psi_1(t) = e^{-\lambda_1 t} \varphi_1 \) in time \( T_{r,R} \) by some control \( p \in L^2(0, T_{r,R}) \)
Global exact controllability outside $\varphi_1$\!

\[
\begin{cases}
    u'(t) + Au(t) + p(t)Bu(t) = 0 & (t > 0) \\
    u(0) = u_0
\end{cases}
\] (S)
Global exact controllability outside $\varphi_1^\perp$

\[
\begin{dcases}
    u'(t) + Au(t) + p(t)Bu(t) = 0 & (t > 0) \\
    u(0) = u_0
\end{dcases}
\] (S)

Corollary

**Suppose there exists a constant $\gamma > 0$ such that the eigenvalues of $A$ fulfill the gap condition**

\[
\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} \geq \gamma \quad \forall k \geq 1
\]

**Let $B : X \to X$ be a linear bounded operator satisfying the following**

\[
\langle B\varphi_1, \varphi_1 \rangle \neq 0 \quad \& \quad \exists b, q > 0 \text{ such that } \lambda_k^q |\langle B\varphi_1, \varphi_k \rangle| \geq b \quad \forall k > 1
\] (⋆⋆)
Global exact controllability outside $\varphi_1$

\[
\left\{ \begin{array}{l}
u'(t) + A u(t) + p(t) B u(t) = 0 \quad (t > 0) \\
u(0) = u_0
\end{array} \right. \tag{S}
\]

Corollary

Suppose there exists a constant $\gamma > 0$ such that the eigenvalues of $A$ fulfill the gap condition

$$\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} \geq \gamma \quad \forall k \geq 1$$

Let $B : X \to X$ be a linear bounded operator satisfying the following

$$\langle B \varphi_1, \varphi_1 \rangle \neq 0 \quad \& \quad \exists b, q > 0 \text{ such that } \lambda_k^q \langle B \varphi_1, \varphi_k \rangle \geq b \quad \forall k > 1 \tag{**}$$

If $\langle u_0, \varphi_1 \rangle \neq 0$, then for every $R > 0$ there exists $T_R > 0$ such that for all $u_0$ satisfying

$$||u_0 - \langle u_0, \varphi_1 \rangle \varphi_1|| \leq R |\langle u_0, \varphi_1 \rangle|$$

the solution to (S) can be steered to $\langle u_0, \varphi_1 \rangle \psi_1$ in time $T_R$ some control $p \in L^2(0, T_R)$
Conclusions

Under the gap condition

\[ \sqrt{\lambda^k} + 1 - \sqrt{\lambda^k} \geq \gamma > 0 \quad \forall k \geq 1 \]

for the eigenvalues of \( A^*A \), we have shown that the control system

\[ u'(t) + A^*Au(t) + p(t)BBBu(t) = 0 \quad (t > 0) \]

\[ u(0) = u_0 \]

is:

• locally superexponentially stabilizable to \( \psi_1(t) = e^{-\lambda_1 t}\chi_1 \)
  provided that

\[ \langle BBB\chi_1\chi_1\chi_1, \chi_k\chi_k\chi_k \rangle \neq 0 \quad \forall k \geq 1 \]

and \( \exists \tau > 0 \) such that

\[ \sum_{k=1}^{\infty} e^{-2\lambda_k \tau} |\langle BBB\chi_1\chi_1\chi_1, \chi_k\chi_k\chi_k \rangle|^2 < \infty \]

(\( \star \))

• locally exactly controllable to \( \psi_1(t) = e^{-\lambda_1 t}\chi_1 \)
  in any time \( T > 0 \) provided that

\[ \langle BBB\chi_1\chi_1\chi_1, \chi_1\chi_1\chi_1 \rangle \neq 0 \]

and \( \exists b, q > 0 \) such that

\[ \lambda^q_k |\langle BBB\chi_1\chi_1\chi_1, \chi_k\chi_k\chi_k \rangle| \geq b \quad \forall k > 1 \]

(\( \star\star \))

This approach can be extended to suitable classes of (unbounded) operators \( BBB \):

\[ D(BBB) \subset X \rightarrow X \]
Conclusions

Under the gap condition

\[ \sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} \geq \gamma > 0 \quad \forall k \geq 1 \]

for the eigenvalues of \( A \), we have shown that the control system

\[
\begin{cases}
  u'(t) + Au(t) + p(t)Bu(t) = 0 \quad (t > 0) \\
  u(0) = u_0
\end{cases}
\]

is:

• locally superexponentially stabilizable to \( \psi_1(t) = e^{-\lambda_1 t} \phi_1 \) provided that \( \langle BBB \phi_1 \phi_1 \phi_1, \phi_k \phi_k \phi_k \rangle \neq 0 \forall k \geq 1 \) & \exists \tau > 0 \text{ such that } \sum_{k=1}^{\infty} e^{-2\lambda_k \tau} |\langle BBB \phi_1 \phi_1 \phi_1, \phi_k \phi_k \phi_k \rangle|^2 < \infty \quad \text{(*)}

• locally exactly controllable to \( \psi_1(t) = e^{-\lambda_1 t} \phi_1 \) in any time \( T > 0 \) provided that \( \langle BBB \phi_1 \phi_1 \phi_1, \phi_1 \phi_1 \phi_1 \rangle \neq 0 \) & \exists b, q > 0 \text{ such that } \lambda_{qk} \geq \frac{b}{\langle BBB \phi_1 \phi_1 \phi_1, \phi_k \phi_k \phi_k \rangle} \forall k > 1 \quad \text{(**)}

This approach can be extended to suitable classes of (unbounded) operators \( BBB : D(BBB) \subset X \to X \)
Conclusions

Under the gap condition

$$\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} \geq \gamma > 0 \quad \forall k \geq 1$$

for the eigenvalues of $A$, we have shown that the control system

$$\begin{cases}
u'(t) + Au(t) + p(t)Bu(t) = 0 \quad (t > 0) \\
u(0) = u_0
\end{cases}$$

is:

- locally superexponentially stabilizable to $\psi_1(t) = e^{-\lambda_1 t}\varphi_1$ provided that

$$\langle B\varphi_1, \varphi_k \rangle \neq 0 \quad \forall k \geq 1 \quad \& \quad \exists \tau > 0 \quad \text{such that} \quad \sum_{k=1}^{\infty} e^{-2\lambda_k \tau} \frac{1}{|\langle B\varphi_1, \varphi_k \rangle|^2} < \infty \quad (\star)$$
Conclusions

Under the gap condition

$$\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} \geq \gamma > 0 \quad \forall k \geq 1$$

for the eigenvalues of $A$, we have shown that the control system

$$\begin{cases} u'(t) + Au(t) + p(t)Bu(t) = 0 \quad (t > 0) \\ u(0) = u_0 \end{cases}$$

is:

- locally superexponentially stabilizable to $\psi_1(t) = e^{-\lambda_1 t} \varphi_1$ provided that

$$\langle B\varphi_1, \varphi_k \rangle \neq 0 \quad \forall k \geq 1 \quad \& \quad \exists \tau > 0 \quad \text{such that} \quad \sum_{k=1}^{\infty} \frac{e^{-2\lambda_k \tau}}{|\langle B\varphi_1, \varphi_k \rangle|^2} < \infty \quad (\star)$$

- locally exactly controllable to $\psi_1(t) = e^{-\lambda_1 t} \varphi_1$ in any time $T > 0$ provided that

$$\langle B\varphi_1, \varphi_1 \rangle \neq 0 \quad \& \quad \exists b, q > 0 \quad \text{such that} \quad \lambda_k^q |\langle B\varphi_1, \varphi_k \rangle| \geq b \quad \forall k > 1 \quad (\star\star)$$

This approach can be extended to suitable classes of (unbounded) operators $BBB$:
Conclusions

Under the gap condition

\[ \sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} \geq \gamma > 0 \quad \forall k \geq 1 \]

for the eigenvalues of \( A \), we have shown that the control system

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\begin{cases}
  u'(t) + Au(t) + p(t)Bu(t) = 0 \quad (t > 0) \\
  u(0) = u_0
\end{cases}
\]

is:

- locally superexponentially stabilizable to \( \psi_1(t) = e^{-\lambda_1 t} \psi_1 \) provided that

  \[ \langle B\varphi_1, \varphi_k \rangle \neq 0 \quad \forall k \geq 1 \quad \& \quad \exists \tau > 0 \text{ such that } \sum_{k=1}^{\infty} e^{-2\lambda_k \tau} < \infty \quad (\star) \]

- locally exactly controllable to \( \psi_1(t) = e^{-\lambda_1 t} \psi_1 \) in any time \( T > 0 \) provided that

  \[ \langle B\varphi_1, \varphi_1 \rangle \neq 0 \quad \& \quad \exists b, q > 0 \text{ such that } \lambda_k^q |\langle B\varphi_1, \varphi_k \rangle| \geq b \quad \forall k > 1 \quad (\star\star) \]

This approach can be extended to suitable classes of (unbounded) operators \( B : D(B) \subset X \to X \).
Best wishes to Jean-Pierre!
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Thank you