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Analysis of control problems of nonmontone semilinear elliptic equations

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A joint work with Mariano Mateos (University of Oviedo, Spain) and Arnd Rösch (University of Duisburg-Essen, Germany)

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The Control Problem

(P)
$$\min_{u \in \mathcal{U}_{ad}} J(u) := \frac{1}{2} \int_{\Omega} (y_u(x) - y_d(x))^2 \, dx + \frac{\nu}{2} \int_{\Omega} u^2(x) \, dx \quad (\nu > 0)$$

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$$\mathcal{U}_{ad} = \{ u \in L^2(\Omega) : \alpha \le u(x) \le \beta \text{ a.e. in } \Omega \}$$
$$(-\infty \le \alpha < \beta \le +\infty)$$





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$$\left\{ \begin{array}{l} Ay+b(x)\cdot\nabla y+f(x,y)=u \text{ in }\Omega\\ y=0 \text{ on }\Gamma \end{array} \right.$$



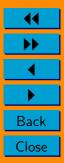


Assumptions on the Linear Operator

• Ω is an open domain in \mathbb{R}^n , n = 2 or 3, with Lipschitz boundary Γ



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$$Ay = -\sum_{i,j=1}^{n} \partial_{x_j}(a_{ij}(x)\partial_{x_i}y)$$
 with $a_{ij} \in L^{\infty}(\Omega)$,

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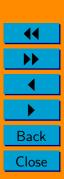
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 with $a_{ij} \in L^{\infty}(\Omega)$,
• $\exists \Lambda > 0$ such that $\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \ge \Lambda |\xi|^2 \quad \forall \xi \in \mathbb{R}^n$ and for a.a. $x \in \Omega$



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- $\bullet \; b \in L^p(\Omega)$ for some p to be fixed later

Study of the Linear Operator

• $\mathcal{A}y = Ay + b(x) \cdot \nabla y + a_0(x)y$, $a_0 \ge 0$



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Study of the Linear Operator

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Theorem. Let $b \in L^p(\Omega)^n$ with p > n, $a_0 \in L^q(\Omega)$ with q > 1 if n = 2 and $q \ge \frac{3}{2}$ if n = 3. Then, $\mathcal{A} : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$ is an isomorphism.







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• If $Ay \le 0 \Rightarrow y \le 0$ (Gilbarg and Trudinger)







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- If $Ay \le 0 \Rightarrow y \le 0$ (Gilbarg and Trudinger)
- We take $0 < \rho < \operatorname{ess\,sup}_{x \in \Omega} y(x)$, $z(x) = (y(x) \rho)^+$.

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- $0 \ge \langle \mathcal{A}z, z \rangle \ge \Lambda \|\nabla z\|_{L^2(\Omega)^3}^2 \|b\|_{L^3(\Omega_\rho)^3} \|\nabla z\|_{L^2(\Omega)^3} \|z\|_{L^6(\Omega)}$



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- $\bullet \ \|b\|_{L^3(\Omega_\rho)^3} \geq \tfrac{C}{\Lambda} > 0 \ \forall \rho$
- Fredholm Alternative



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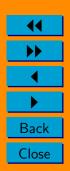
Regularity of the Solution

Theorem. Assume that $a_{ij} \in C^{0,1}(\overline{\Omega})$ for $1 \leq i, j \leq n$, $b \in L^p(\Omega)^n$ for some p > n, and $a_0 \in L^2(\Omega)$. We also suppose that Γ is of class $C^{1,1}$ or Ω is convex. Then, $\mathcal{A} : H^2(\Omega) \cap H^1_0(\Omega) \longrightarrow L^2(\Omega)$ is an isomorphism.



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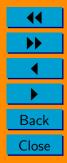
The Adjoint Equation

Corollary. The adjoint operator $\mathcal{A}^* : H^1_0(\Omega) \longrightarrow H^{-1}(\Omega)$ given by

 $\mathcal{A}^*\varphi = A^*\varphi - \operatorname{div}[b(x)\varphi] + a_0(x)\varphi$

is an isomorphism.





The Adjoint Equation

Corollary. The adjoint operator $\mathcal{A}^* : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$ given by

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Corollary. Assume that $a_{ij} \in C^{0,1}(\overline{\Omega})$ for $1 \leq i, j \leq n, b \in L^p(\Omega)^n$ for some p > n, and $a_0 \in L^2(\Omega)$. We also suppose that $\operatorname{div} b \in L^2(\Omega)$, Γ is of class $C^{1,1}$ or Ω is convex. Then, $\mathcal{A}^* : H^2(\Omega) \cap H^1_0(\Omega) \longrightarrow L^2(\Omega)$ is an isomorphism.



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Proof.
$$\mathcal{A}^* \varphi = A^* \varphi - b(x) \nabla \varphi + (a_0(x) - \operatorname{div} b(x)) \varphi$$

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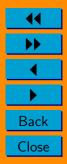
Assumptions on Semilinear Equation

• $f: \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, monotone nondecreasing with respect to the second variable



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Assumptions on Semilinear Equation

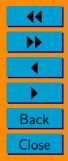
• $f: \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, monotone nondecreasing with respect to the second variable

• $\forall M > 0 \ \exists \phi_M \in L^{\bar{p}}(\Omega) \text{ with } \bar{p} > \frac{n}{2} : |f(x,y)| \le \phi_M(x)$ for a.a. $x \in \Omega$ and $\forall |y| \le M$



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Theorem. Let $b \in L^p(\Omega)^n$ with p > 2 if n = 2 and p > 6 if n = 3. For every $u \in L^{\bar{p}}(\Omega)$ the semilinear equation has a unique solution y_u in $H_0^1(\Omega) \cap C(\bar{\Omega})$.



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$$\|y_u\|_{H^1_0(\Omega)} + \|y_u\|_{C(\bar{\Omega})} \le K_f \Big(\|u\|_{L^{\bar{p}}(\Omega)} + \|f(\cdot, 0)\|_{L^{\bar{p}}(\Omega)} + 1\Big)$$

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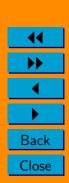
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A Monotonicity Result

Lemma. Under the assumptions of the above theorem, if $y_1, y_2 \in H_0^1(\Omega) \cap C(\overline{\Omega})$ are solutions of the equations

 $Ay_i + b(x) \cdot \nabla y_i + f(x, y_i) = u_i, \quad i = 1, 2,$

with $u_1, u_2 \in L^{\bar{p}}(\Omega)$ and $u_1 \leq u_2$ in Ω , then $y_1 \leq y_2$ in Ω as well.



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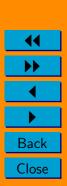
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with $u_1, u_2 \in L^{\bar{p}}(\Omega)$ and $u_1 \leq u_2$ in Ω , then $y_1 \leq y_2$ in Ω as well.

• The uniqueness of a solution of the state equation is consequence of the above lemma.



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Existence: Sketch of Proof

• We redefine $f = f - f(\cdot, 0)$ and $u = u - f(\cdot, 0) \in L^{\bar{p}}(\Omega)$.



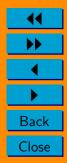
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 $f_k(x,y) = f(x,\min\{y,k\})$







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 $f_k(x,y) = f(x,\min\{y,k\}) \Rightarrow \phi(x) \le f_k(x,y) \le \phi_k(x)$



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$$\begin{aligned} f_k(x,y) &= f(x,\min\{y,k\}) \Rightarrow \phi(x) \le f_k(x,y) \le \phi_k(x) \\ Ay_k + b(x) \cdot \nabla y_k + f_k(x,y_k) = u \\ Ay + b(x) \cdot \nabla y = u - \phi \end{aligned}$$



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$$A(y - y_k) + b(x) \cdot \nabla (y - y_k) = f_k(x, y_k) - \phi \ge 0$$



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$$f_k(x, y) = f(x, \min\{y, k\}) \Rightarrow \phi(x) \le f_k(x, y) \le \phi_k(x)$$

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$$\Rightarrow y_k \le y \Rightarrow f_k(x, y_k) = f(x, y_k) \text{ if } k \ge \|y\|_{\infty}$$



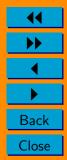
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• Step 3: The general case.

 $f_k(x,y) = f(x, \operatorname{proj}_{[-k,+k]}(y))$



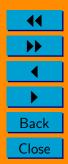


• Step 3: The general case.

$$f_k(x, y) = f(x, \operatorname{proj}_{[-k, +k]}(y))$$

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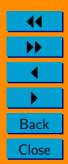
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• Step 3: The general case.

$$\begin{aligned} f_k(x,y) &= f(x, \operatorname{proj}_{[-k,+k]}(y)) \\ Ay_k + b(x) \cdot \nabla y_k + f_k(x,y_k) &= u \\ Az_1 + b(x) \cdot \nabla z_1 + f(x,z_1^+) &= u \end{aligned}$$





• Step 3: The general case.

$$\begin{split} f_k(x,y) &= f(x, \text{proj}_{[-k,+k]}(y)) \\ Ay_k + b(x) \cdot \nabla y_k + f_k(x, y_k) &= u \\ Az_1 + b(x) \cdot \nabla z_1 + f(x, z_1^+) &= u \\ Az_1 + b(x) \cdot \nabla z_1 + f_k(x, z_1) &= u + f_k(x, z_1) - f(x, z_1^+) \leq u \end{split}$$



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• Step 3: The general case.

$$\begin{split} f_k(x,y) &= f(x, \text{proj}_{[-k,+k]}(y)) \\ Ay_k + b(x) \cdot \nabla y_k + f_k(x, y_k) &= u \\ Az_1 + b(x) \cdot \nabla z_1 + f(x, z_1^+) &= u \\ Az_1 + b(x) \cdot \nabla z_1 + f_k(x, z_1) &= u + f_k(x, z_1) - f(x, z_1^+) \leq u \\ &\Rightarrow z_1 \leq y_k \quad \forall k \geq 1 \end{split}$$



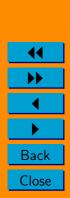
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Image: A triangle of the sector of

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Continuity of $u \to y$ and regularity of y





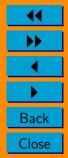
Continuity of $u \to y$ and regularity of y

Theorem. Let $\{u_k\}_{k=1}^{\infty} \subset L^{\bar{p}}(\Omega)$ with $\bar{p} > \frac{n}{2}$ be a sequence weakly converging to u in $L^{\bar{p}}(\Omega)$. Then, $y_{u_k} \to y_u$ strongly in $H_0^1(\Omega) \cap C(\bar{\Omega})$, where y_{u_k} is the solution of the semilinear equation associated to u_k .



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Continuity of $u \to y$ and regularity of y

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Theorem. Suppose that assumption on f holds with $\bar{p} = 2$, $a_{ij} \in C^{0,1}(\bar{\Omega})$ for $1 \leq i, j \leq n$, and $b \in L^p(\Omega)^n$ with p > 2 if n = 2 and p > 6 if n = 3. We also suppose that Γ is of class $C^{1,1}$ or Ω is convex. Then, for every $u \in L^2(\Omega)$ the state equation has a unique solution $y_u \in H^2(\Omega) \cap H^1_0(\Omega)$.



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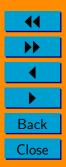
Existence of solution of (P)

 \bullet We recall that

(P)
$$\min_{u \in \mathcal{U}_{ad}} J(u) := \frac{1}{2} \int_{\Omega} (y_u(x) - y_d(x))^2 \, dx + \frac{\nu}{2} \int_{\Omega} u^2(x) \, dx \quad (\nu > 0)$$



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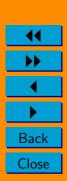


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Theorem. The control problem (P) has at least one solution \bar{u} .



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Differentiability Assumptions on f





Differentiability Assumptions on f

• $f: \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is of class C^2 w.r.t. the second variable





Differentiability Assumptions on f

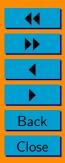
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$$f(\cdot,0) \in L^{\bar{p}}(\Omega)$$
 with $\bar{p} > \frac{n}{2}$ and $\frac{\partial f}{\partial y}(x,y) \ge 0$ a.e. in Ω and $\forall y \in \mathbb{R}$

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•
$$\forall M > 0 \ \exists C_{f,M} : \left| \frac{\partial f}{\partial y}(x,y) \right| + \left| \frac{\partial^2 f}{\partial y^2}(x,y) \right| \le C_{f,M} \ \forall |y| \le M$$

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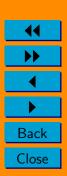
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•
$$\left\{ \begin{array}{l} \forall M > 0 \text{ and } \forall \varepsilon > 0 \ \exists \delta > 0 \text{ such that} \\ \left| \frac{\partial^2 f}{\partial y^2}(x, y_2) - \frac{\partial^2 f}{\partial y^2}(x, y_1) \right| < \varepsilon \text{ if } |y_1|, |y_2| \le M, \ |y_2 - y_1| \le \delta \end{array} \right.$$



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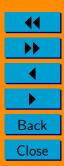
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Differentiability of the Mapping $u \to y_u$

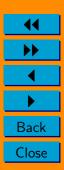




Differentiability of the Mapping $u \to y_u$

Given $\hat{p} > \frac{n}{2}$, let us denote $G : L^{\hat{p}}(\Omega) \longrightarrow Y = H^1_0(\Omega) \cap C(\overline{\Omega})$ the mapping associating with each control the state $G(u) = y_u$.





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Theorem. The control-to-state mapping G is of class C^2



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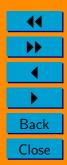
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Theorem. The control-to-state mapping G is of class C^2 and for every $u, v \in L^{\hat{p}}(\Omega)$, we have that $z_v = G'(u)v$ is the solution of

$$\begin{cases} Az + b(x) \cdot \nabla z + \frac{\partial f}{\partial y}(x, y_u)z = v \text{ in } \Omega \\ z = 0 \text{ on } \Gamma \end{cases}$$







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and for $v,w \in L^{\hat{p}}(\Omega)$, $z_{v,w} = G''(u)(v,w)$ solves the equation

$$\begin{cases} Az + b(x) \cdot \nabla z + \frac{\partial f}{\partial y}(x, y_u)z + \frac{\partial^2 f}{\partial y^2}(x, y_u)z_v z_w = 0 \text{ in } \Omega\\ z = 0 \text{ on } \Gamma \end{cases}$$



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Analysis of the Cost Functional

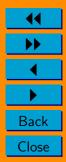




Analysis of the Cost Functional

Theorem. The functional $J: L^2(\Omega) \longrightarrow \mathbb{R}$ is of class C^2 .





Analysis of the Cost Functional

Theorem. The functional $J: L^2(\Omega) \longrightarrow \mathbb{R}$ is of class C^2 . Moreover, given $u, v, v_1, v_2 \in L^2(\Omega)$ we have

$$J'(u)v = \int_{\Omega} (\varphi_u + \nu u)v \, dx$$
$$J''(u)(v_1, v_2) = \int_{\Omega} \left[1 - \varphi_u \frac{\partial^2 f}{\partial y^2}(x, y_u) \right] z_{v_1} z_{v_2} \, dx + \nu \int_{\Omega} v_1 v_2 \, dx$$

where $\varphi_u \in H^1_0(\Omega) \cap C(\bar{\Omega})$ is the unique solution of the adjoint equation

$$\begin{cases} A^* \varphi - \operatorname{div}[b(x)\varphi] + \frac{\partial f}{\partial y}(x, y_u)\varphi = y_u - y_d \text{ in } \Omega, \\ \varphi = 0 \text{ on } \Gamma. \end{cases}$$



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Local Solutions





Local Solutions

Definition. We say that $\bar{u} \in U_{ad}$ is an $L^r(\Omega)$ -weak local minimum of (P), with $r \in [1, +\infty]$, if there exists some $\varepsilon > 0$ such that

 $J(\bar{u}) \leq J(u) \quad \forall u \in \mathcal{U}_{ad} \text{ with } \|\bar{u} - u\|_{L^{r}(\Omega)} \leq \varepsilon.$



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An element $\bar{u} \in \mathcal{U}_{ad}$ is said a strong local minimum of (P) if there exists some $\varepsilon > 0$ such that

$$J(\bar{u}) \leq J(u) \quad \forall u \in \mathcal{U}_{ad} \text{ with } \|y_{\bar{u}} - y_u\|_{L^{\infty}(\Omega)} \leq \varepsilon.$$

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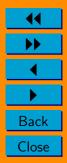
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We say that $\bar{u} \in \mathcal{U}_{ad}$ is a strict (weak or strong) local minimum if the above inequalities are strict for $u \neq \bar{u}$.

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I - Relationships among these Notions





I - Relationships among these Notions

 \bullet If \mathcal{U}_{ad} is bounded in $L^2(\Omega)\text{, then}$



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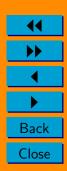


I - Relationships among these Notions

- \bullet If \mathcal{U}_{ad} is bounded in $L^2(\Omega),$ then
 - 1. \bar{u} is an $L^1(\Omega)$ -weak local minimum of (P) if and only if it is an $L^r(\Omega)$ -weak local minimum of (P) for every $r \in (1, +\infty)$.



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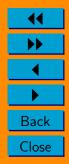
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 - 2. If \bar{u} is a $L^r(\Omega)$ -weak local minimum of (P) for some $r < +\infty$, then it is a $L^{\infty}(\Omega)$ -weak local minimum of (P).



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 - 3. \bar{u} is a strong local minimum of (P) if and only if it is an $L^r(\Omega)$ -weak local minimum of (P) for all $r \in [1, \infty)$.



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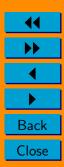


II - Relationships among these Notions



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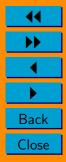
II - Relationships among these Notions

• If \mathcal{U}_{ad} is not bounded in $L^2(\Omega)$, then



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II - Relationships among these Notions

- If \mathcal{U}_{ad} is not bounded in $L^2(\Omega)$, then
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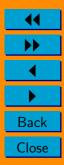


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 - 3. \bar{u} is an $L^2(\Omega)\text{-weak}$ local solution if and only if it is a strong local solution.

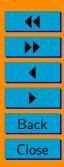


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First Order Optimality Conditions





First Order Optimality Conditions

Theorem. Let \bar{u} be a local solution of (P) in any of the previous senses, then there exist two unique elements $\bar{y}, \bar{\varphi} \in H_0^1(\Omega) \cap C(\bar{\Omega})$ such that

$$\begin{cases} A\bar{y} + b(x) \cdot \nabla \bar{y} + f(x, \bar{y}) = \bar{u} \text{ in } \Omega, \\ \bar{y} = 0 \text{ on } \Gamma, \end{cases} \\ \begin{cases} A^* \bar{\varphi} - \operatorname{div}[b(x)\bar{\varphi}] + \frac{\partial f}{\partial y}(x, \bar{y})\bar{\varphi} = \bar{y} - y_d \text{ in } \Omega, \\ \bar{\varphi} = 0 \text{ on } \Gamma, \end{cases} \\ \int_{\Omega} (\bar{\varphi} + \nu \bar{u})(u - \bar{u}) \, dx \ge 0 \quad \forall u \in \mathcal{U}_{ad} \end{cases}$$







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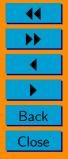
$$\begin{cases} A\bar{y} + b(x) \cdot \nabla \bar{y} + f(x, \bar{y}) = \bar{u} \text{ in } \Omega, \\ \bar{y} = 0 \text{ on } \Gamma, \end{cases} \\ \begin{cases} A^* \bar{\varphi} - \operatorname{div}[b(x)\bar{\varphi}] + \frac{\partial f}{\partial y}(x, \bar{y})\bar{\varphi} = \bar{y} - y_d \text{ in } \Omega, \\ \bar{\varphi} = 0 \text{ on } \Gamma, \end{cases} \\ \int_{\Omega} (\bar{\varphi} + \nu \bar{u})(u - \bar{u}) \, dx \ge 0 \quad \forall u \in \mathcal{U}_{ad} \end{cases}$$

Moreover, $\bar{u} \in H^1(\Omega) \cap C(\bar{\Omega})$ holds.



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Second Order Conditions





Second Order Conditions

•Cone of critical directions:

$$C_{\bar{u}} = \left\{ v \in L^2(\Omega) : J'(\bar{u})v = 0 \text{ and } v(x) \left\{ \begin{array}{l} \geq 0 \text{ if } \bar{u}(x) = \alpha \\ \leq 0 \text{ if } \bar{u}(x) = \beta \end{array} \right\}$$



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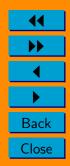
Second Order Conditions

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Theorem If \bar{u} is a local minimum of (P), then $J''(\bar{u})v^2 \ge 0 \ \forall v \in C_{\bar{u}}$.





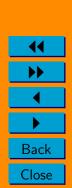
Second Order Conditions

•Cone of critical directions:

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$$J(\bar{u}) + \frac{\kappa}{2} \|u - \bar{u}\|_2^2 \le J(u) \quad \forall u \in \mathcal{U}_{ad} : \|y_u - \bar{y}\|_{L^{\infty}(Q)} \le \varepsilon.$$



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Second Order Conditions

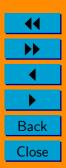
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Approximation of the state equation



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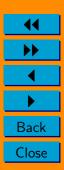
Approximation of the state equation

• We consider a quasi-uniform family of triangulations $\{\mathcal{T}_h\}_{h>0}$ of $\overline{\Omega}$.



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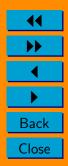
 $Y_h = \{ y_h \in C(\overline{\Omega}) : y_{h|T} \in P_1(T) \ \forall T \in \mathcal{T}_h \text{ and } y_h \equiv 0 \text{ on } \Gamma \}.$



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$$\begin{aligned} a(y_1, y_2) &= \langle \mathcal{A}y_1, y_2 \rangle_{H^{-1}(\Omega), H^1_0(\Omega)} \\ &= \int_{\Omega} \left(\sum_{i,j=1}^n a_{ij}(x) \partial_{x_i} y_1 \partial_{x_j} y_2 + [b(x) \cdot \nabla y_1] y_2 \right) dx. \end{aligned}$$



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Approximation of the state equation

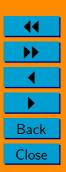
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=
$$\int_{\Omega} \left(\sum_{i,j=1}^n a_{ij}(x) \partial_{x_i} y_1 \partial_{x_j} y_2 + [b(x) \cdot \nabla y_1] y_2 \right) dx.$$

 $\begin{cases} \text{Find } y_h \in Y_h \text{ such that} \\ a(y_h, z_h) + \int_{\Omega} f(x, y_h(x)) z_h(x) \, dx = \int_{\Omega} u(x) z_h(x) \, dx \ \, \forall z_h \in Y_h. \end{cases}$



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Numerical Analysis of the Linear Equation





Numerical Analysis of the Linear Equation

Theorem [Schatz, 1974]. Let $a_0 \in L^2(\Omega)$ be a nonnegative function. There exists $h_A > 0$ depending on \mathcal{A} and $||a_0||_{L^2(\Omega)}$ such that the variational problem

$$\begin{cases} \text{Find } y_h \in Y_h \text{ such that} \\ a(y_h, z_h) + \int_{\Omega} a_0(x) y_h(x) z_h(x) \, dx = \int_{\Omega} u(x) z_h(x) \, dx \quad \forall z_h \in Y_h \end{cases}$$

has a unique solution for every $h \leq h_A$ and for every $u \in L^2(\Omega)$. Moreover, there exists a constant C_{A,a_0} such that

$$\|y_h\|_{H^1_0(\Omega)} \le C_{\mathcal{A},a_0} \|\mathcal{A}^{-1}u\|_{H^1_0(\Omega)} \quad \forall h \le h_{\mathcal{A}}.$$

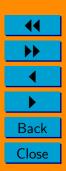








Numerical Analysis of the Semilinear Equation



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Numerical Analysis of the Semilinear Equation

Theorem. Let us assume that

 $\left\{ \begin{array}{l} f(\cdot,0)\in L^2(\Omega) \text{ and } \forall M>0 \ \exists L_{f,M} \text{ such that} \\ |f(x,y_2)-f(x,y_1)|\leq L_{f,M}|y_2-y_1| \ \forall |y_i|\leq M, \ i=1,2. \end{array} \right.$

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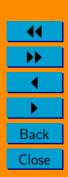
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 $\forall M \geq 1 + \|y\|_{C(\bar{\Omega})} \exists h_M > 0$ such that for every $h < h_M$ the discrete equation has a unique solution y_h satisfying $\|y_h\|_{C(\bar{\Omega})} \leq M$.



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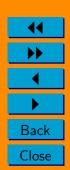
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$$\|y - y_h\|_{L^2(\Omega)} + h \|y - y_h\|_{H^1_0(\Omega)} \le K_M \Big(\|u\|_{L^2(\Omega)} + 1 \Big) h^2 \|y - y_h\|_{L^\infty(\Omega)} \le K_{\infty,M} \Big(\|u\|_{L^2(\Omega)} + 1 \Big) h^{2-\frac{n}{2}}$$



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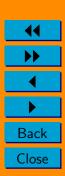
$$\|y - y_h\|_{L^2(\Omega)} + h\|y - y_h\|_{H^1_0(\Omega)} \le K_M \Big(\|u\|_{L^2(\Omega)} + 1\Big)h^2 \|y - y_h\|_{L^\infty(\Omega)} \le K_{\infty,M} \Big(\|u\|_{L^2(\Omega)} + 1\Big)h^{2-\frac{n}{2}}$$

Further, if there exist other solutions $\{\tilde{y}_h\}_{h < h_M}$ with $y_h \neq \tilde{y}_h$ for all h, then $\lim_{h \to 0} \|\tilde{y}_h\|_{C(\bar{\Omega})} = \infty$.

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A Discrete Mapping $u_h \rightarrow y_h$



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A Discrete Mapping $u_h \to y_h$

Theorem. Let $\bar{y} \in Y$ be the solution of state equation corresponding to the control $\bar{u} \in L^2(\Omega)$. Given $\rho > 0$ arbitrary, there exist $\rho^* > 0$ and $h_0 > 0$ such that the discrete equation has a unique solution $y_h(u) \in \bar{B}_{\rho^*}^Y(\bar{y})$ for every $u \in \bar{B}_{\rho}(\bar{u}) \subset L^2(\Omega)$ and for all $h < h_0$, where

$$B_{\rho^*}^Y(\bar{y}) = \{ y \in Y : \|y - \bar{y}\|_Y \le \rho^* \}.$$

Furthermore, there exist constants K and K_∞ such that

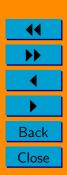
$$||y_u - y_h(u)||_{L^2(\Omega)} + h||y_u - y_h(u)||_{H^1_0(\Omega)} \le K \Big(||\bar{u}||_{L^2(\Omega)} + \rho + 1 \Big) h^2 ||y_u - y_h(u)||_{L^\infty(\Omega)} \le K_\infty \Big(||\bar{u}||_{L^2(\Omega)} + \rho + 1 \Big) h^{2-\frac{n}{2}} \quad \forall u \in \bar{B}\rho(\bar{u})$$

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Numerical Approximation of (P)



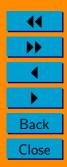
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Numerical Approximation of (P)

 \bullet Let us define $\mathcal{J}:L^2(\Omega)\times L^2(\Omega)\to \mathbb{R}$ given by

$$\mathcal{J}(y,u) = \frac{1}{2} \int_{\Omega} (y(x) - y_d(x))^2 dx + \frac{\nu}{2} \int_{\Omega} u^2 dx$$



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• Let us denote by \mathcal{U}_h one of the following two spaces:

$$\mathcal{U}_{h} = \mathcal{U}_{h}^{0} := \{ u_{h} \in L^{2}(\Omega) : u_{h|T} \in P_{0}(T) \ \forall T \in \mathcal{T}_{h} \}$$
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- We set $\mathcal{U}_{h,ad} = \mathcal{U}_h \cap \mathcal{U}_{ad}$.
- \bullet We approximate Problem (P) by the problem

 $(\mathcal{P}_h) \min \{ \mathcal{J}(y_h, u_h) : (y_h, u_h) \in Y_h imes \mathcal{U}_{h,ad} \text{ satisfies the discrete equation} \}$

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Convergence of (\mathcal{P}_h) to (P)



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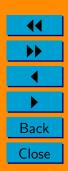
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Convergence of (\mathcal{P}_h) to (P)

Theorem. There exists $h_0 > 0$ such that problem (\mathcal{P}_h) has at least one solution (\bar{y}_h, \bar{u}_h) for all $h < h_0$.

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Convergence of (\mathcal{P}_h) to (P)

Theorem. There exists $h_0 > 0$ such that problem (\mathcal{P}_h) has at least one solution (\bar{y}_h, \bar{u}_h) for all $h < h_0$. Moreover, if $\{(\bar{y}_h, \bar{u}_h)\}_{h < h_0}$ is a sequence of solutions of problems (\mathcal{P}_h) , then it is bounded in $H_0^1(\Omega) \times L^2(\Omega)$ and there exist subsequences converging weakly in $H_0^1(\Omega) \times L^2(\Omega)$.



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Error Estimates





Error Estimates

Theorem. Let $\bar{u} \in L^2(\Omega)$ be a local minimizer of (P) satisfying the sufficient second order optimality conditions and let $\{\bar{u}_h\}$ be the sequence of minimizers of the problems (\mathcal{P}_h) described in the above theorem. Then, there exists $h_0 > 0$ such that

• If $\mathcal{U}_{ad} \subsetneq L^2(\Omega)$, then

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)} \le Ch. \quad \forall h < h_0$$

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• If $\mathcal{U}_{ad} = L^2(\Omega)$ and $\mathcal{U}_h = \mathcal{U}_h^i$, i = 0, 1, then

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)} \le Ch^{1+i} \quad \forall h < h_0$$

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THANK YOU VERY MUCH FOR YOUR ATTENTION

