Dedicated to Jean-Pierre Raymond

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University of Cantabria

Analysis of control problems of nonmontone semilinear elliptic equations

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A joint work with Mariano Mateos (University of Oviedo, Spain) and Arnd Rösch (University of Duisburg-Essen, Germany)

The Control Problem

$$(\mathrm{P}) \qquad \min_{u \in \mathcal{U}_{ad}} J(u) := \frac{1}{2} \int_\Omega (y_u(x) - y_d(x))^2 \, dx + \frac{\nu}{2} \int_\Omega u^2(x) \, dx \quad (\nu > 0)$$

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$$\begin{cases} Ay + b(x)\cdot\nabla y + f(x,y) = u \text{ in } \Omega \\ y = 0 \text{ on } \Gamma \end{cases}$$

Back
Close

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- $\exists \Lambda > 0$ such that $\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \Lambda|\xi|^2 \ \ \forall \xi \in \mathbb{R}^n$ and for a.a. $x \in \Omega$

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- $b \in L^p(\Omega)$ for some p to be fixed later

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- $0 \geq \langle \mathcal{A}z, z\rangle \geq \Lambda \|\nabla z\|^2_{L^2(\Omega)^3} - \|b\|_{L^3(\Omega_\rho)^3} \|\nabla z\|_{L^2(\Omega)^3} \|z\|_{L^6(\Omega)}$

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- Fredholm Alternative

Regularity of the Solution

Theorem. Assume that $a_{ij} \in C^{0,1}(\bar{\Omega})$ for $1 \leq i,j \leq n$, $b \in L^p(\Omega)^n$ for some $p > n$, and $a_0 \in L^2(\Omega)$. We also suppose that Γ is of class $C^{1,1}$ or Ω is convex. Then, $\mathcal{A} : H^2(\Omega) \cap H_0^1(\Omega) \longrightarrow L^2(\Omega)$ is an isomorphism.

The Adjoint Equation

Corollary. The adjoint operator $\mathcal{A}^* : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$ given by

$$\mathcal{A}^*\varphi = A^*\varphi - \mathrm{div}[b(x)\varphi] + a_0(x)\varphi$$

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Proof. $\mathcal{A}^*\varphi = A^*\varphi - b(x)\nabla\varphi + (a_0(x) - \mathrm{div}\, b(x))\varphi$

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$$\|y_u\|_{H_0^1(\Omega)} + \|y_u\|_{C(\bar\Omega)} \le K_f \Big(\|u\|_{L^{\bar p}(\Omega)} + \|f(\cdot,0)\|_{L^{\bar p}(\Omega)} + 1 \Big)$$

A Monotonicity Result

Lemma. Under the assumptions of the above theorem, if $y_1, y_2 \in H_0^1(\Omega) \cap C(\bar{\Omega})$ are solutions of the equations

$$Ay_i + b(x) \cdot \nabla y_i + f(x, y_i) = u_i, \quad i = 1, 2,$$

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• The uniqueness of a solution of the state equation is consequence of the above lemma.

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& \Rightarrow y_k \le y \Rightarrow f_k(x,y_k) = f(x,y_k) \ \text{ if } \ k \ge \|y\|_\infty
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Continuity of $u \to y$ and regularity of y

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Theorem. Let $\{u_k\}_{k=1}^{\infty} \subset L^{\bar{p}}(\Omega)$ with $\bar{p} > \frac{n}{2}$ be a sequence weakly converging to u in $L^{\bar{p}}(\Omega)$. Then, $y_{u_k} \to y_u$ strongly in $H_0^1(\Omega) \cap C(\bar{\Omega})$, where y_{u_k} is the solution of the semilinear equation associated to u_k.

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Theorem. Suppose that assumption on f holds with $\bar{p} = 2$, $a_{ij} \in C^{0,1}(\bar{\Omega})$ for $1 \leq i, j \leq n$, and $b \in L^p(\Omega)^n$ with $p > 2$ if $n = 2$ and $p > 6$ if $n = 3$. We also suppose that Γ is of class $C^{1,1}$ or Ω is convex. Then, for every $u \in L^2(\Omega)$ the state equation has a unique solution $y_u \in H^2(\Omega) \cap H_0^1(\Omega)$.

Existence of solution of (P)

- We recall that

$$(\mathrm{P}) \qquad \min_{u \in \mathcal{U}_{ad}} J(u) := \frac{1}{2}\int_\Omega (y_u(x) - y_d(x))^2\,dx + \frac{\nu}{2}\int_\Omega u^2(x)\,dx \quad (\nu > 0)$$

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Theorem. The control problem (P) has at least one solution $\bar{u}$.

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- $\forall M > 0\ \exists C_{f,M} : \left|\dfrac{\partial f}{\partial y}(x, y)\right| + \left|\dfrac{\partial^2 f}{\partial y^2}(x, y)\right| \leq C_{f,M}\ \forall |y| \leq M$

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- $\left\{ \begin{array}{l} \forall M > 0 \text{ and } \forall \varepsilon > 0\ \exists \delta > 0 \text{ such that} \\ \left|\frac{\partial^2 f}{\partial y^2}(x, y_2) - \frac{\partial^2 f}{\partial y^2}(x, y_1)\right| < \varepsilon \text{ if } |y_1|, |y_2| \leq M,\ |y_2 - y_1| \leq \delta \end{array} \right.$

Differentiability of the Mapping $u \to y_u$

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Given $\hat{p} > \frac{n}{2}$, let us denote $G : L^{\hat{p}}(\Omega) \longrightarrow Y = H_0^1(\Omega) \cap C(\bar{\Omega})$ the mapping associating with each control the state $G(u) = y_u$.

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Theorem. The control-to-state mapping G is of class C^2 and for every $u, v \in L^{\hat{p}}(\Omega)$, we have that $z_v = G'(u)v$ is the solution of

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and for $v, w \in L^{\hat{p}}(\Omega)$, $z_{v,w} = G''(u)(v, w)$ solves the equation

$$\begin{cases} Az + b(x) \cdot \nabla z + \dfrac{\partial f}{\partial y}(x, y_u)z + \dfrac{\partial^2 f}{\partial y^2}(x, y_u)z_v z_w = 0 \text{ in } \Omega \\ z = 0 \text{ on } \Gamma \end{cases}$$

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Theorem. The functional $J : L^2(\Omega) \longrightarrow \mathbb{R}$ is of class C^2. Moreover, given $u, v, v_1, v_2 \in L^2(\Omega)$ we have

$$J'(u)v = \int_\Omega (\varphi_u + \nu u)v \, dx$$

$$J''(u)(v_1, v_2) = \int_\Omega \left[1 - \varphi_u \frac{\partial^2 f}{\partial y^2}(x, y_u)\right] z_{v_1} z_{v_2} \, dx + \nu \int_\Omega v_1 v_2 \, dx$$

where $\varphi_u \in H_0^1(\Omega) \cap C(\bar{\Omega})$ is the unique solution of the adjoint equation

$$\begin{cases} A^*\varphi - \mathrm{div}[b(x)\varphi] + \dfrac{\partial f}{\partial y}(x, y_u)\varphi = y_u - y_d \text{ in } \Omega, \\ \varphi = 0 \text{ on } \Gamma. \end{cases}$$

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Local Solutions

Local Solutions

Definition. We say that $\bar{u} \in \mathcal{U}_{ad}$ is an $L^r(\Omega)$-weak local minimum of (P), with $r \in [1, +\infty]$, if there exists some $\varepsilon > 0$ such that

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An element $\bar{u} \in \mathcal{U}_{ad}$ is said a strong local minimum of (P) if there exists some $\varepsilon > 0$ such that

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We say that $\bar{u} \in \mathcal{U}_{ad}$ is a strict (weak or strong) local minimum if the above inequalities are strict for $u \neq \bar{u}$.

I - Relationships among these Notions

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3. $\bar{u}$ is an $L^2(\Omega)$-weak local solution if and only if it is a strong local solution.

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First Order Optimality Conditions

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Theorem. Let $\bar{u}$ be a local solution of (P) in any of the previous senses, then there exist two unique elements $\bar{y}, \bar{\varphi} \in H_0^1(\Omega) \cap C(\bar{\Omega})$ such that

$$\begin{cases} A\bar{y} + b(x) \cdot \nabla \bar{y} + f(x, \bar{y}) = \bar{u} \text{ in } \Omega, \\ \bar{y} = 0 \text{ on } \Gamma, \end{cases}$$

$$\begin{cases} A^*\bar{\varphi} - \operatorname{div}[b(x)\bar{\varphi}] + \dfrac{\partial f}{\partial y}(x, \bar{y})\bar{\varphi} = \bar{y} - y_d \text{ in } \Omega, \\ \bar{\varphi} = 0 \text{ on } \Gamma, \end{cases}$$

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$$\int_\Omega (\bar{\varphi} + \nu \bar{u})(u - \bar{u})\, dx \geq 0 \quad \forall u \in \mathcal{U}_{ad}$$

Moreover, $\bar{u} \in H^1(\Omega) \cap C(\bar{\Omega})$ holds.

Second Order Conditions

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•Cone of critical directions:

$$C_{\bar{u}} = \left\{ v \in L^2(\Omega) : J'(\bar{u})v = 0 \text{ and } v(x) \left\{ \begin{array}{l} \geq 0 \text{ if } \bar{u}(x) = \alpha \\ \leq 0 \text{ if } \bar{u}(x) = \beta \end{array} \right\} \right.$$

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Theorem If $\bar u$ is a local minimum of (P), then $J''(\bar u)v^2 \geq 0\ \forall v \in C_{\bar u}$. Conversely, if $\bar u \in \mathcal{U}_{ad}$ satisfies the first order optimality conditions and $J''(\bar u)v^2 > 0\ \forall v \in C_{\bar u} \setminus \{0\}$, then there exist $\varepsilon > 0$ and $\kappa > 0$ such that

$$J(\bar u) + \frac{\kappa}{2}\|u - \bar u\|_2^2 \leq J(u) \quad \forall u \in \mathcal{U}_{ad} : \|y_u - \bar y\|_{L^\infty(Q)} \leq \varepsilon.$$

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Theorem Let $\bar u \in \mathcal{U}_{ad}$. Then $J''(\bar u)v^2 > 0 \ \forall v \in C_{\bar u} \setminus \{0\}$ if and only if there exists $\delta > 0$ such that $J''(\bar u)v^2 \geq \delta \|v\|^2_{L^2(\Omega)} \ \forall v \in C_{\bar u}$.

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$$\left\{ \begin{array}{l} \text{Find } y_h \in Y_h \text{ such that} \\ a(y_h, z_h) + \displaystyle\int_\Omega f(x, y_h(x)) z_h(x) \, dx = \int_\Omega u(x) z_h(x) \, dx \ \ \forall z_h \in Y_h. \end{array} \right.$$

Numerical Analysis of the Linear Equation

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Theorem [Schatz, 1974]. Let $a_0 \in L^2(\Omega)$ be a nonnegative function. There exists $h_{\mathcal{A}} > 0$ depending on $\mathcal{A}$ and $\|a_0\|_{L^2(\Omega)}$ such that the variational problem

$$\begin{cases} \text{Find } y_h \in Y_h \text{ such that} \\ a(y_h, z_h) + \displaystyle\int_\Omega a_0(x) y_h(x) z_h(x)\, dx = \int_\Omega u(x) z_h(x)\, dx \quad \forall z_h \in Y_h \end{cases}$$

has a unique solution for every $h \le h_{\mathcal{A}}$ and for every $u \in L^2(\Omega)$. Moreover, there exists a constant $C_{\mathcal{A},a_0}$ such that

$$\|y_h\|_{H_0^1(\Omega)} \le C_{\mathcal{A},a_0} \|\mathcal{A}^{-1} u\|_{H_0^1(\Omega)} \quad \forall h \le h_{\mathcal{A}}.$$

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Theorem. Let us assume that

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$$\|y - y_h\|_{L^2(\Omega)} + h\|y - y_h\|_{H_0^1(\Omega)} \le K_M\Big(\|u\|_{L^2(\Omega)} + 1\Big)h^2$$

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$$\|y - y_h\|_{L^2(\Omega)} + h\|y - y_h\|_{H_0^1(\Omega)} \leq K_M\Big(\|u\|_{L^2(\Omega)} + 1\Big)h^2$$

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Further, if there exist other solutions $\{\tilde y_h\}_{h<h_M}$ with $y_h \neq \tilde y_h$ for all h, then $\lim_{h\to 0}\|\tilde y_h\|_{C(\bar\Omega)} = \infty$.

A Discrete Mapping $u_h \to y_h$

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Theorem. Let $\bar{y} \in Y$ be the solution of state equation corresponding to the control $\bar{u} \in L^2(\Omega)$. Given $\rho > 0$ arbitrary, there exist $\rho^* > 0$ and $h_0 > 0$ such that the discrete equation has a unique solution $y_h(u) \in \bar{B}^Y_{\rho^*}(\bar{y})$ for every $u \in \bar{B}_\rho(\bar{u}) \subset L^2(\Omega)$ and for all $h < h_0$, where

$$B^Y_{\rho^*}(\bar{y}) = \{y \in Y : \|y - \bar{y}\|_Y \leq \rho^*\}.$$

Furthermore, there exist constants K and K_∞ such that

$$\|y_u - y_h(u)\|_{L^2(\Omega)} + h\|y_u - y_h(u)\|_{H^1_0(\Omega)} \leq K\Big(\|\bar{u}\|_{L^2(\Omega)} + \rho + 1\Big)h^2$$
$$\|y_u - y_h(u)\|_{L^\infty(\Omega)} \leq K_\infty\Big(\|\bar{u}\|_{L^2(\Omega)} + \rho + 1\Big)h^{2-\frac{n}{2}} \quad \forall u \in \bar{B}\rho(\bar{u})$$

Numerical Approximation of (P)

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- Let us define $\mathcal{J} : L^2(\Omega) \times L^2(\Omega) \to \mathbb{R}$ given by

$$\mathcal{J}(y,u) = \frac{1}{2}\int_\Omega (y(x) - y_d(x))^2\,dx + \frac{\nu}{2}\int_\Omega u^2\,dx$$

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- Let us denote by $\mathcal{U}_h$ one of the following two spaces:

$$\mathcal{U}_h = \mathcal{U}_h^0 := \{u_h \in L^2(\Omega) : \ u_{h|T} \in P_0(T) \ \forall T \in \mathcal{T}_h\}$$

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- We set $\mathcal{U}_{h,ad} = \mathcal{U}_h \cap \mathcal{U}_{ad}$.
- We approximate Problem (P) by the problem

$(\mathcal{P}_h)$ $\min\{\mathcal{J}(y_h, u_h) :\ (y_h, u_h) \in Y_h \times \mathcal{U}_{h,ad}$ satisfies the discrete equation$\}$.

Convergence of $(\mathcal{P}_h)$ to (P)

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Convergence of $(\mathcal{P}_h)$ to (P)

Theorem. There exists $h_0 > 0$ such that problem $(\mathcal{P}_h)$ has at least one solution $(\bar{y}_h, \bar{u}_h)$ for all $h < h_0$.

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Theorem. There exists $h_0 > 0$ such that problem $(\mathcal{P}_h)$ has at least one solution $(\bar{y}_h, \bar{u}_h)$ for all $h < h_0$. Moreover, if $\{(\bar{y}_h, \bar{u}_h)\}_{h<h_0}$ is a sequence of solutions of problems $(\mathcal{P}_h)$, then it is bounded in $H_0^1(\Omega) \times L^2(\Omega)$ and there exist subsequences converging weakly in $H_0^1(\Omega) \times L^2(\Omega)$.

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Error Estimates

Error Estimates

Theorem. Let $\bar{u} \in L^2(\Omega)$ be a local minimizer of (P) satisfying the sufficient second order optimality conditions and let $\{\bar{u}_h\}$ be the sequence of minimizers of the problems $(\mathcal{P}_h)$ described in the above theorem. Then, there exists $h_0 > 0$ such that

• If $\mathcal{U}_{ad} \subsetneq L^2(\Omega)$, then

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)} \leq Ch. \quad \forall h < h_0$$

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• If $\mathcal{U}_{ad} \subsetneq L^2(\Omega)$, then

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• If $\mathcal{U}_{ad} = L^2(\Omega)$ and $\mathcal{U}_h = \mathcal{U}_h^i$, $i = 0, 1$, then

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)} \leq Ch^{1+i} \quad \forall h < h_0$$

Dedicated to Jean-Pierre Raymond

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THANK YOU VERY MUCH FOR YOUR ATTENTION