Controllability of the Wave Equation with Rough Coefficients

Part I [1]

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Setting

$$(W) \qquad \begin{cases} \partial_t^2 u - \Delta_x u = 0 \qquad \text{in} \quad]0,+\infty[\times M \\ (u(0), \partial_t u(0)) = (u_0, u_1) \in H^1 \times L^2 \end{cases}$$

- M Riemannian manifold, connected, compact, without boundary, with dimension d.
- $M = \Omega$ open subset of $\mathbb{R}^d$, connected, bounded, with "smooth" boundary (homogeneous Dirichlet condition).

$$H = C([0,+\infty[, H^1) \cap C^1([0,+\infty[, L^2)$$

$\longrightarrow$ Consider ω an open subset of M and Γ an open subset of $\partial\Omega$ and also a time $T > 0$.

The Goal

Provide internal or boundary exact control results in the case of **non smooth metrics**.

Given (u_0, u_1), find a control vector f (resp. g) s.t the solution of

$$\begin{cases} \partial_t^2 u - \Delta_x u = \chi_\omega f \\ (u(0), \partial_t u(0)) = (u_0, u_1) \end{cases}$$

resp.

$$\begin{cases} \partial_t^2 u - \Delta_x u = 0 \\ u = \chi_\Gamma g \qquad \text{on} \quad \partial\Omega \\ (u(0), \partial_t u(0)) = (u_0, u_1) \end{cases}$$

satisfies $u(T) = \partial_t u(T) = 0$.

Tool : By HUM , we need an **observability estimate** for the wave equation (W).

Observability estimates

Boundary observation

$$Eu(0) \leq c \int_0^T \int_\Gamma \left| \frac{\partial u}{\partial n}(t,x) \right|^2 d\sigma dt$$

Remark: The converse is "always" true :

$$\int_0^T \int_{\partial\Omega} \left| \frac{\partial u}{\partial n}(t,x) \right|^2 d\sigma dt \leq c\, Eu(0)$$

$\rightarrow$ Hidden regularity.

Internal observation

$$Eu(0) \leq c \int_0^T \int_\omega |\partial_t u(t,x)|^2 dxdt \qquad (O)$$

Or at least

$$Eu(0) \leq c \int_0^T \int_\omega |\partial_t u(t,x)|^2 dxdt + c||(u_0, u_1)||^2_{L^2(M)\times H^{-1}(M)} \qquad (RO)$$

$\rightarrow$ unique continuation property....

Or either observation with loss

$$Eu(0) \leq c||u||^2_{H^m((0,T)\times\omega)}, \qquad m > 1 \qquad (OL)$$

Remark : If M is a compact manifold without boundary, we consider instead the Klein-Gordon equation

$$\partial_t^2 u - \Delta_x u + u = 0$$

Applications

$\rightarrow$ **Exact controllability** (HUM)

$\rightarrow$ **Stabilization**

$$Eu(t) \leq C\, exp^{-\gamma t} Eu(0)$$

for solutions of the damped equation

$$\partial_t^2 u - \Delta_x u + a(x)\partial_t u = 0$$

$\rightarrow$ **Inverse problems**

Stability results, ...

State of the art

- 80' : Observability estimates under the Γ-condition of J.L. Lions.
  $\rightarrow$ Metric of class C^1.
  $\rightarrow$ Multiplier techniques.

- 90' : Microlocal conditions and microlocal tools (Rauch and Taylor, Bardos, Lebeau and Rauch, Burq and Gérard).
  The geometric control condition **(G.C.C)** : a microlocal condition, stated in the cotangent bundle (of Melrose-Sjostrand).

  $\rightarrow$ Microlocal and pseudo-differential techniques : propagation of wave front sets and supports of microlocal defect measures.
  $\rightarrow$ The condition is optimal but....... a priori needs smooth metric and smooth boundary.

- 97' N. Burq : Boundary observability: C^2-metric and C^3-boundary.

Geometric Control Condition I

The couple (ω, T) satisfies the geometric control condition (G.C.C), if every geodesic of Ω issued at $t = 0$ and travelling with speed 1, enters in ω before the time T.

Geometric Control Condition II

The couple (Γ, T) satisfies the geometric control condition (G.C.C), if every generalized bicharacteristic of the wave symbol, issued at $t = 0$, intersects the boundary subset Γ at a nondiffractive point, before the time T.

Some Functional Spaces

$a(x)$ is resp. Zygmund or Log-Zygmund continuous function if

$|h| < 1$,

$$\begin{cases} |a(x+h) + a(x-h) - 2a(x)| \leq K|h| \\ |a(x+h) + a(x-h) - 2a(x)| \leq K|h|(1 - Log|h|) \end{cases}$$

For $0 < \alpha < 1$,

$$W^{1,\infty} = Lip \subset Z = C^1_* \subset LL \subset LZ \subset C^\alpha$$

The OL theorem of Fanelli-Zuazua (2014)

1-D setting, $\Omega =]0, 1[$, $a(x)\partial_t^2 u - \partial_x^2 u = 0.$

Theorem (Fanelli - Zuazua 1)

Assume $a(x) \in Z$, *then for* $T > T_a$, *there exists* $C > 0$ *s.t*

$$Eu(0) \leq C \int_0^T |\partial_x u(t, 0)|^2 dt \tag{1}$$

Theorem (Fanelli - Zuazua 2)

Assume $a(x) \in LZ$ *and denote* $D_a f = a(x)^{-1}\partial_x^2 f$. *Then for* $T > T_a$, *there exist* $C > 0$ *and* $m \in \mathbb{N}$ *s.t*

$$Eu(0) \leq C \int_0^T |\partial_t^m \partial_x u(t, 0)|^2 dt \tag{2}$$

for all initial data $(u_0, u_1) \in (H^{2m+1} \cap H_0^1) \times H^{2m}$ *satisfying*

$$D_a^m u_0 \in H^1 \qquad D_a^m u_1 \in L^2$$

Comments

- Classical boundary observation in Th. 1 and boundary observation with **loss** in Th. 2.

- For $a(x)$ worse that LZ, infinite loss of derivatives :
  No Observability !
  See also the counter-example of Castro - Zuazua (03').

- Proof: 1-dimensional technique: the sidewise energy estimates, i.e hyperbolic energy estimates by interchanging time $\longleftrightarrow$ space.
  (Colombini, Spagnolo, Lerner, Métivier, Fanelli....)

- 1-D geometry: all characteristic rays reach the boundary in uniform time.

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- 1-D geometry: all characteristic rays reach the boundary in uniform time.
- **Question**: What about dimensions higher than 1, where geometry is more evolved ???

Case of a continuous density $a(x)$ in $\mathbb{R}^d$

$$(W) \qquad \begin{cases} a(x)\partial_t^2 u - \Delta_x u = 0 & \text{in } (0,T)\times\Omega, \\ u = 0 & \text{on } (0,T)\times\partial\Omega, \\ (u(0),\partial_t u(0)) \in H_0^1 \times L^2 & \end{cases}$$

Assumption : There exists $\alpha \in (0,2]$, s.t

$$x\cdot\nabla a(x) + (2-\alpha)a(x) \geq 0 \quad \text{in the sense of } \mathcal{D}'(\mathbb{R}^d).$$

$$\forall\varphi \in D(\mathbb{R}^d), \text{ with } \varphi \geq 0, \quad \int_{\mathbb{R}^d} a(x)\left(-div(x\varphi(x)) + (2-\alpha)\varphi(x)\right)\, dx \geq 0$$

Observation region

ω an open neighborhood (in Ω) of an open subset Γ of the boundary satisfying the multiplier condition, i.e

$$\{x \in \partial\Omega, \quad \text{such that} \quad x \cdot n_x > 0\} \subset \Gamma,$$

Let $R = sup\{|x|,\, x \in \Omega\}$ and $a_1 = sup\{a(x),\, x \in \overline{\Omega}\}$

Theorem (D-Ervedoza 17')

For $\alpha T > 4R\sqrt{a_1}$, there exists a constant $C > 0$ s.t the observability estimate

$$Eu(0) \leq C \int_0^T \int_\omega |\partial_t u(t,x)|^2 dxdt$$

holds true for every solution of (W).

Examples

- One can take $a(x) = a(r,\theta) = f(r)g(\theta)$
  with f, g positive continuous, and $f'(r) \geq 0$ in the sense of distributions.

  $\rightarrow$ Allows highly oscillating function $g(\theta)$.

- $\Omega = B(0,R) \setminus B(0,R_1)$, $0 < R_1 < R$, and $a(x) = 1/r^2$.

  Take $x_0 \in \Omega$, $\xi_0 \neq 0$, $x_0.\xi_0 = 0$, and $\tau_0 = |x_0||\xi_0|$

  The ray $\gamma(s)$ issued from $(0, x_0, \tau_0, \xi_0)$ satisfies

$$\frac{d^2}{ds^2}\left(|x(s)|^2\right) = 0 \quad \text{and} \quad \frac{d}{ds}\left(|x(s)|^2\right)\Big|_{s=0} = 0$$

  $\rightarrow$ $|x(s)| = |x_0|$ Captive ray !

Stability properties

Denote $A(x) = (a_{ij}(x))$, $d \times d$ symmetric definite positive matrix, and κ a real valued function, $\kappa(x) > 0$ (a density).

Denote $\mathcal{A} = (A, \kappa)$,

$$\Delta_{\mathcal{A}} = \frac{1}{\kappa(x)} \sum_{ij} \partial_j a_{ij}(x) \kappa(x) \partial_i$$

and consider the wave operator

$$P_{\mathcal{A}} = \partial_t^2 - \Delta_{\mathcal{A}}$$

Theorem (Burq-D-Le Rousseau 19')

Assume that $\mathcal{A} = (A, \kappa)$ is smooth and that (ω, T) (resp. (Γ, T)) satisfies (GCC) for $P_{\mathcal{A}}$.
Take $\mathcal{B} \in \mathcal{U}_\varepsilon$, an ε-neighborhood of $\mathcal{A}$ in $W^{1,\infty}$.
Then for ε **small enough**, *the (classical) observability estimate holds true*

$$Eu(0) \leq c \int_0^T \int_\omega |\partial_t u(t,x)|^2 \kappa dx dt \quad \left(resp. \int_0^T \int_\Gamma \left|\frac{\partial u}{\partial n}(t,x)\right|^2 \kappa d\sigma dt \right)$$

for every solution of

$$P_{\mathcal{B}} u = \partial_t^2 u - \Delta_{\mathcal{B}} u = 0, \qquad u_{|\partial\Omega} = 0$$

Corollary

Under conditions above, we get exact controllability for $P_{\mathcal{B}} = \partial_t^2 - \Delta_{\mathcal{B}}$, in time T.

Comments

→ No geometry setting for the metric B !!!

→ One can replace $Eu(0)$ by $E_B u(0)$.

→ One can also consider the same problem on a perturbed domain Ω_ε, with $W^{2,\infty}$ perturbation.

$$\text{metric} \quad A \longrightarrow B = A_\varepsilon \qquad W^{1,\infty} \quad \varepsilon - \text{perturbation}$$

$$\text{domain} \quad \Omega \longrightarrow \Omega_\varepsilon \qquad W^{2,\infty} \quad \varepsilon - \text{perturbation}$$

→ For a given metric $B \in W^{1,\infty}$, we cannot decide if $P_{\mathcal{B}}$ is observable or not.
In particular, what happens for $B \in C^1$ and $\partial\Omega$ of class C^2 ???

Next lecture by Jérôme !

Behavior of the HUM control process

$$(W) \quad \begin{cases} \partial_t^2 u - \Delta_{\mathcal{A}} u = \chi_\omega^2(x) f & \text{in} \quad]0,T[\times\Omega \\ u = 0 & \text{on} \quad]0,T[\times\partial\Omega \\ (u(0), \partial_t u(0)) = (u_0, u_1) \in H_0^1 \times L^2 \end{cases}$$

We look for $f \in L^2(]0,T[\times\Omega)$, s.t

$$(u(T), \partial_t u(T)) = (0,0)$$

By HUM and **under** (**G.C.C**), we can take f solution of

$$(W') \quad \begin{cases} \partial_t^2 f - \Delta_{\mathcal{A}} f = 0 & \text{in} \quad]0,T[\times\Omega \\ f = 0 & \text{on} \quad]0,T[\times\partial\Omega \\ (f(0), \partial_t f(0)) = (f_0, f_1) \in L^2 \times H^{-1} \end{cases}$$

The map

$$\begin{cases} \Lambda : H_0^1 \times L^2 \to L^2 \times H^{-1} \\ \\ (u_0, u_1) \to (f_0, f_1) \end{cases}$$

is an isomorphism; this is HUM optimal control operator.

Denote by $f_{\mathcal{A}}$ the HUM control attached to $P_{\mathcal{A}}$, i.e the solution of (W').

Theorem (D-Lebeau, 2009)

In the setting above and under (G.C.C),
a) For all $s \geq 0$,

$$\Lambda : H^{s+1} \times H^s \to H^s \times H^{s-1}$$

is an isomorphism.

b)

$$\left\| \Lambda \psi(2^{-k} D) - \psi(2^{-k} D) \Lambda \right\| \leq C 2^{-k/2}$$

c) If M *is a Riemannian manifold without boundary,* Λ *is a pseudo differential operator.*

Behavior of the HUM control process

Let $\mathcal{A} = (A, \kappa)$ be smooth (C^2) and such that (ω, T) satisfies (GCC).

Theorem (Burq-D-Le Rousseau 19')

For any C^2- neighborhood $\mathcal{U}$ of $\mathcal{A}$, there exist $\mathcal{A}' \in \mathcal{U}$ and an initial data (u_0, u_1) , $||(\nabla_A u_0, u_1)||_{L^2 \times L^2} = 1$, s.t the respective solutions u and v of

$$\begin{cases} \partial_t^2 u - \Delta_{\mathcal{A}} u = \chi_\omega^2(x) f_{\mathcal{A}} \\ \partial_t^2 v - \Delta_{\mathcal{A}'} v = \chi_\omega^2(x) f_{\mathcal{A}} \\ (u(0), \partial_t u(0)) = (v(0), \partial_t v(0)) = (u_0, u_1) \in H_0^1 \times L^2 \end{cases}$$

satisfy

$$E_A(u - v)(T) = E_A(v)(T) \geq 1/2$$

Moreover,

$$||f_{\mathcal{A}} - f_{\mathcal{A}'}||_{L^2((0,T)\times\omega)} \geq 1/\sqrt{8T}$$

Remarks

- (GCC) also satisfied by (ω, T) for the metric $\mathcal{A}'$.

  $\rightarrow \quad f_{\mathcal{A}'}$ is well defined.

- For fixed initial data, the map

$$(\mathcal{C}^2(\Omega))^{d^2+1} \longrightarrow L^2((0,T)\times\omega)$$

$$\mathcal{A} = (A, \kappa) \longrightarrow f_{\mathcal{A}}$$

  is not continuous.

Proof

$\rightarrow$ Choose $A' = (1+\varepsilon)A$

$\rightarrow$ Take a sequence (u_0^k, u_1^k) such that $||(\nabla u_0^k, u_1^k)||_{L^2\times L^2} = 1$ and $(u_0^k, u_1^k) \rightharpoonup (0,0)$ in $H_0^1 \times L^2$, .

$\rightarrow$ $f_{\mathcal{A}}^k \rightharpoonup 0$ in $L^2((0,T)\times\Omega)$. Hence $f_{\mathcal{A}}^k \to 0$ in $H^{-1}((0,T)\times\Omega)$ and $\mathrm{supp}\mu(\mathrm{f}_{\mathcal{A}}^k) \subset Char(\partial_t^2 - \Delta_{\mathcal{A}}) = \{\tau^2 - A_x(\xi,\xi) = 0\}$.

$\rightarrow$ $\mathrm{supp}\mu(\mathrm{v}^k) \subset Char(\partial_t^2 - \Delta_{\mathcal{A}'}) = \{\tau^2 - A'_x(\xi,\xi) = 0\}$

$\rightarrow$ $\mathrm{supp}\mu(\mathrm{v}^k) \cap supp\mu(\mathrm{f}_{\mathcal{A}}^k) = \varnothing$

$$E_{A'}(v^k)(T) - E_{A'}(v^k)(0) = 2\int_0^T\int_\Omega \chi_\omega^2(x) f_{\mathcal{A}}^k \partial_t v^k \, dxdt \longrightarrow 0.$$

On the other hand, decompose this function v as $v = v_1 + v_2$ where

$$\begin{cases} \Box_{\mathcal{A}'} v_1 = \chi_\omega^2(x) f_{\mathcal{A}'} & (v_1(0), \partial_t v_1(0)) = (u_0, u_1) \\ \Box_{\mathcal{A}'} v_2 = \chi_\omega^2(x)(f_{\mathcal{A}} - f_{\mathcal{A}'}) & (v_2(0), \partial_t v_2(0)) = (0, 0) \end{cases}$$

Clearly $(v_1(T), \partial_t v_1(T)) = (0,0)$, hence $E_{A'}(v)(T) = E_{A'}(v_2)(T)$

$\longrightarrow$ Conclude with hyperbolic energy estimate.

Theorem (Control of smooth data)

Assume that (ω, T) *satisfies* (GCC) *for the metric* A*. Then for any metric* B *of class* C^1*, and any* $\alpha \in]0,1]$*, the respective solutions* u *and* v *of*

$$\begin{cases} P_A u = \chi_\omega^2(x) f_A & in \quad (0,T)\times\Omega \\ P_B v = \chi_\omega^2(x) f_A & in \quad (0,T)\times\Omega \\ u = v = 0 & on \quad (0,T)\times\partial\Omega \\ (u(0),\partial_t u(0)) = (v(0),\partial_t v(0)) = (u_0,u_1) \in H^{1+\alpha}(\Omega)\times H^\alpha(\Omega) \end{cases}$$

satisfy

$$E_A^{1/2}(u-v)(T) \le c_\alpha ||A-B||_{C^1}^\alpha \times ||(u_0,u_1)||_{H^{1+\alpha}\times H^\alpha}$$

for some constant $c_\alpha > 0$*.*

Proof of the observability estimate

Strategy

→ First prove

$$Eu(0) \leq c \int_0^T \int_\omega |\partial_t u(t,x)|^2 dxdt + ||(u_0, u_1)||^2_{L^2 \times H^{-1}}$$

→ In the smooth case :

Contradiction argument and propagation of micro local defect measures.

Smooth case

→ Contradiction argument

$$\begin{cases} Eu^k(0) = 1, \quad (u^k) \text{ solution of (W)}, \\ \int_0^T \int_\omega |\partial_t u^k(t,x)|^2 dxdt + ||(u_0^k, u_1^k)||^2_{L^2(M)\times H^{-1}(M)} \leq 1/k \end{cases}$$

$$\begin{cases} u^k \rightharpoonup 0 \quad \text{in} \quad H^1((0,T)\times M), \\ \int_0^T \int_\omega |\partial_t u^k(t,x)|^2 dxdt \to 0 \end{cases}$$

Let μ be a **microlocal defect measure** attached to u_k .

→ $\mu = 0$ over $(0,T)\times\omega$ and by propagation and GCC, $\mu = 0$ everywhere.

Contradiction with $Eu^k(0) = 1$.

Back to non smooth metric

To be achieved : Prove a propagation result for μ, in a low regularity setting.

Stability result : Boundaryless case (Warm up)

Contradiction argument

Recall : $A(x) = (a_{ij}(x))$, $d \times d$ symmetric definite positive matrix, and κ a real valued function, $\kappa(x) > 0$ (a density).

Denote $\mathcal{A} = (A, \kappa)$,

$$\Delta_{\mathcal{A}} = \frac{1}{\kappa(x)} \sum_{ij} \partial_j a_{ij}(x) \kappa(x) \partial_i$$

and consider the Klein-Gordon equation :

$$Pu = \partial_t^2 u - \Delta_{\mathcal{A}} u + u = 0 \qquad \text{on} \quad (0, T) \times \Omega$$

$\longrightarrow$ Consider a sequence of $\mathcal{A}_k = (A_k, \kappa_k)$ s.t

$$\|\mathcal{A} - \mathcal{A}_k\|_{W^{1,\infty}} \to 0, \quad \text{and for each } k, \quad \|(u_0^{k,p}, u_1^{k,p})\|_{H^1 \times L^2} = 1$$

$$\begin{cases} P_k u_k^p = \partial_t^2 u_k^p - \Delta_{\mathcal{A}_k} u_k^p + u_k^p = 0 & \text{on} \quad (0,T) \times \Omega \\ u_k^p(0) = u_0^{k,p}, \qquad \partial_t u_k^p(0) = u_1^{k,p} & \end{cases}$$

and

$$\int_0^T \int_\omega |\partial_t u_k^p|^2 dxdt \leq 1/p.$$

Denote $u_k^k \rightsquigarrow u_k$,

$$\begin{cases} P_k u_k = \partial_t^2 u_k - \Delta_{\mathcal{A}_k} u_k + u_k = 0 & \text{on} \quad (0, T) \times \Omega \\ \|(u_0^k, u_1^k)\|_{H^1 \times L^2} = 1 \\ \int_0^T \int_\omega |\partial_t u_k|^2 dxdt \longrightarrow 0 \end{cases}$$

and assume

$$u_k \rightharpoonup 0 \quad \text{weakly in} \quad H^1((0, T) \times \Omega)$$

Let μ be a microlocal defect measure attached to the sequence u_k.

$$P_0 u_k = (P_0 - P_k)u_k = (\Delta_{\mathcal{A}_k} - \Delta_{\mathcal{A}})u_k \longrightarrow 0 \quad \text{in} \quad H^{-1}$$

Consider $Q = q(t, x; D_t, D_x) \in Op(S^1_{cl})$.
And calculate the bracket

$$\Big([P_0, Q]u_k, u_k\Big)_{L^2} = \Big([\Delta_{\mathcal{A}_k} - \Delta_{\mathcal{A}}, Q]u_k, u_k\Big)_{L^2} + o(1/k)$$

.

Theorem (Calderon, Coifman-Meyer 85')

For any function $m \in W^{1,\infty}(\mathbb{R}^{d+1})$, the bracket $[m, Q]$ continuously maps $L^2(\mathbb{R}^{d+1})$ in itself and

$$\|[m, Q]\|_{L^2 \to L^2} \leq C\|m\|_{W^{1,\infty}}$$

Thus $\Big([P_0, Q]u_k, u_k\Big)_{L^2} \longrightarrow 0$ and ${}^t H_{p_0}\mu = 0$.
μ is invariant along the hamiltonian flow of P_0 (propagation).

If we look carefully, the previous bracket calculus also works for a referential metric of class C^1.

This gives

$$ {}^{t}H_{p_0}\mu = 0 $$

where the vector field H_{p_0} has continuous coefficients.

$\longrightarrow$ How to deal with this measure equation ?

Next lecture by Jérôme !