Wave equation and nonlinear damping controls

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Control and stabilization issues for PDE
dedicated to Jean-Pierre Raymond
Page 63: Natural frequency with "good and bad vibrations"

[David A. Lind et Scott P. Sanders, The Physics of Skiing: Skiing at the Triple Point, 2nd edition; 2013]
One way to kill bad vibrations?

Control your skis!

As Jean-Pierre?

Use passively controls
[L. Rothemann, H. Schretter, Active vibration damping of the alpine ski; 2010]

How to do it actively?
Need to control a PDE, with finite energy, that is with saturating controls.

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As Jean-Pierre would do!
Given a PDE, there exists now a large variety on methods to design linear controllers. It is well known that saturation can reduce the performance or even destabilize the system, even for finite-dimensional systems. More precisely, even if

\[ \dot{z} = Az + BKz \]  

(1)

is asymptotically stable, it may hold that

\[ \dot{z} = Az + \text{sat}(BKz) \]  

(2)

is not globally asymptotically stable.

It may exist new equilibrium, new limit cycles...

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See e.g. [Tarbouriech, Garcia, Gomes da Silva Jr., Queinnec; 2011]
Saturating a stabilizing feedback law can lead to \textbf{instabilities}.

An illustrative example

\[
\frac{dz}{dt} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} z + \begin{bmatrix} 0 \\ -1 \end{bmatrix} u
\]

Open-loop eigenvalues: \( \lambda_1 = 1, \lambda_2 = -1 \). Setting \( u = Kz \) with \( K = \begin{bmatrix} 13 & 7 \end{bmatrix} \), the origin is globally asymptotically stable.

Considering \( u = \text{sat}(Kz) \) with saturation level \( u_s = 5 \), we get

1. \( z_0 = [\begin{array}{c} -2 \\ -3 \end{array}]^\top \): the trajectory converges to \( z^* = [\begin{array}{c} -5 \\ 0 \end{array}]^\top \);
2. \( z_0 = [\begin{array}{c} -3 \\ -3 \end{array}]^\top \): the trajectory diverges.
Stability issues: a finite-dimensional example

Figure: (*) initial conditions, (o) equilibrium points
Goal of this talk:
What happens if in (2), instead of matrices $A, B\ldots$, we have operators? More precisely, what happens if $A$ generates a semigroup and $B$ is a bounded control operator? An example of such a nonlinear PDE given by (2):
Wave equation with saturating in-domain control

Two objectives
- Well-posedness
- Stability

of the wave equation in presence of a disturbed saturating control with a Lyapunov method.

[Harraux; 18], [Martinez; 99], [Martinez and Vancostenoble; 00], [Alabau-Boussouira; 12]
1 Well-posedness and stability of linear wave equation with a saturated in-domain control
   Lyapunov method, LaSalle invariance principle

2 Design of a strict Lyapunov function for $L^2$ saturation
   Robustness result

3 With localized ($L^\infty$) saturation
   strict Lyapunov method, robustness result

4 With non-monotone damping
   comparison with a linear time-varying equation

5 Conclusion
Outline

1. Well-posedness and stability of linear wave equation with a saturated in-domain control
   Lyapunov method, LaSalle invariance principle
2. Design of a strict Lyapunov function for $L^2$ saturation
   Robustness result
3. With localized ($L^\infty$) saturation
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1D wave equation with in-domain control.

Dynamics of the vibration:

\[ z_{tt}(x, t) = z_{xx}(x, t) + f(x, t), \ \forall x \in (0, 1), \ t \geq 0, \quad (3) \]

Boundary conditions, \( \forall t \geq 0, \)

\[ z(0, t) = 0, \]
\[ z(1, t) = 0, \quad (4) \]

and with the following initial condition, \( \forall x \in (0, 1), \)

\[ z(x, 0) = z^0(x), \]
\[ z_t(x, 0) = z^1(x), \quad (5) \]

where \( z^0 \) and \( z^1 \) stand respectively for the initial deflection and the initial deflection speed.
When closing the loop with a linear control

Let us define the linear control by

\[ f(x, t) = -az_t(x, t), \quad x \in (0, 1), \quad \forall t \geq 0, \tag{6} \]

and consider the energy

\[ E = \frac{1}{2} \int (z_x^2 + z_t^2) dx. \]

Formal computation. Along the solutions to (3), (4) and (6):

\[
\dot{E} = \int_0^1 (z_x z_{xt} - a z_t^2 + z_t z_{xx}) dx \\
= - \int_0^1 a z_t^2 dx + [z_t z_x]_{x=1}^{x=0} \\
= - \int_0^1 a z_t^2 dx
\]

Thus, it \( a > 0 \), \( E \) is a (non strict) Lyapunov function.
Using standard technics (Lumer-Philipps thereom (for the well-posedness) and Huang-Prüss theorem (for the exp. stability)):

**Proposition**

∀ \( a > 0 \), ∀ \((z^0, z^1)\) in \( H := H^1_0(0, 1) \times L^2(0, 1) \),
∃ ! solution \((z, z_t)\): \([0, \infty) \rightarrow H\) to (3)-(6). Moreover, ∃ \( C, \mu > 0 \), such that, for any initial condition \( H \), it holds, \( \forall t \geq 0 \),

\[
\|z\|_{H^1_0(0,1)} + \|z_t\|_{L^2(0,1)} \leq Ce^{-\mu t}(\|z^0\|_{H^1_0(0,1)} + \|z^1\|_{L^2(0,1)}).
\]

In the previous proposition:

- stability
- attractivity of the equilibrium
- with an exponential speed
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$\forall a > 0, \forall (z^0, z^1) \text{ in } H := H^1_0(0, 1) \times L^2(0, 1),$

$\exists \text{ ! solution } (z, z_t): [0, \infty) \to H \text{ to (3)-(6). Moreover, } \exists C, \mu > 0, \text{ such that, for any initial condition } H, \text{ it holds, } \forall t \geq 0,$

$$\|z\|_{H^1_0(0,1)} + \|z_t\|_{L^2(0,1)} \leq Ce^{-\mu t}(\|z^0\|_{H^1_0(0,1)} + \|z^1\|_{L^2(0,1)}).$$

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\[ \forall a > 0, \forall (z^0, z^1) \text{ in } H := H^1_0(0,1) \times L^2(0,1), \]

\[ \exists ! \text{ solution } (z, z_t): [0, \infty) \to H \text{ to (3)-(6)}. \]

Moreover, \[ \exists C, \mu > 0, \]

\[ \text{such that, for any initial condition } H, \text{ it holds, } \forall t \geq 0, \]

\[ \|z\|_{H^1_0(0,1)} + \|z_t\|_{L^2(0,1)} \leq Ce^{-\mu t}(\|z^0\|_{H^1_0(0,1)} + \|z^1\|_{L^2(0,1)}). \]

In the previous proposition:

- stability
- attractivity of the equilibrium
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When closing the loop with a saturating control

Let us consider now the nonlinear control

\[ f(x, t) = -\text{sat}(az_t(x, t)), \ x \in (0, 1), \ \forall t \geq 0, \tag{7} \]

where \( \text{sat} \) is the localized saturated map:

\[
\text{sat}(\sigma) = \begin{cases} 
\sigma & \text{if } |\sigma| < 1 \\
\text{sign}(\sigma) & \text{else}
\end{cases}
\]

Equation (3) in closed loop with the control (7) becomes

\[ z_{tt} = z_{xx} - \text{sat}(az_t) \tag{8} \]

A formal computation gives, along the solutions to (8) and (4),

\[ \dot{E} = - \int_0^1 z_t \text{sat}(az_t) \, dx \]

which asks to handle the nonlinearity \( z_t \text{sat}(az_t) \).
Remark: Choice of the saturation map

[Slemrod; 89] and [Lasiecka, Seidman; 03] deal with $L^2$ saturation:
Given $\sigma : [0, 1] \to \mathbb{R}$, $\text{sat}_2(\sigma)$ is the function defined by

$$\text{sat}_2(\sigma)(x) = \begin{cases} 
\sigma(x) & \text{if } \|\sigma\|_{L^2(0,1)} < 1 \\
\frac{\sigma(x)}{\|\sigma\|_{L^2(0,1)}} & \text{else}
\end{cases}$$

Here we consider localized saturation which is more physically relevant:

$$\text{sat}(\sigma(x)) = \begin{cases} 
\sigma(x) & \text{if } |\sigma(x)| < 1 \\
\text{sign}(\sigma(x)) & \text{else}
\end{cases}$$
Difference between both saturations maps

$$\text{sat}_2(\sigma)(x) = \begin{cases} 
\sigma(x) & \text{if } \|\sigma\|_{L^2(0,1)} < 1 \\
\frac{\sigma(x)}{\|\sigma\|_{L^2(0,1)}} & \text{else}
\end{cases}$$

$$\text{sat}(\sigma(x)) = \begin{cases} 
\sigma(x) & \text{if } |\sigma(x)| < 1 \\
\text{sign}(\sigma(x)) & \text{else}
\end{cases}$$

with $\sigma = 2\cos$
Well-posedness of this nonlinear PDE

Theorem 1 [CP, Tarbouriech, Gomes da Silva Jr; 2016]

∀a ≥ 0, for all (z^0, z^1) in (H^2(0, 1) ∩ H_0^1(0, 1)) × H_0^1(0, 1), there exists a unique (strong) solution z: [0, ∞) → H^2(0, 1) ∩ H_0^1(0, 1) to (8) with the boundary conditions (4) and the initial condition (5).
Consider

\[ A_1 \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ u_{xx} - \text{sat}(av) \end{pmatrix} \]

with the domain \( D(A_1) = (H^2(0, 1) \cap H^1_0(0, 1)) \times H^1_0(0, 1) \).

Let us use a generalization of Lumer-Phillips theorem which is the so-called Crandall-Liggett theorem, as given in [Barbu; 1976]. See also [Brezis; 1973] and [Miyadera; 1992].

Again two conditions

1. \( A_1 \) is dissipative, that is

\[ \Re \left( \langle A_1 \begin{pmatrix} u \\ v \end{pmatrix} - A_1 \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} \rangle \right)_H \leq 0 \]

2. For all \( \lambda > 0 \), \( D(A_1) \subseteq \text{Ran}(I - \lambda A_1) \)
First item: Easy step!

Instead of proving
\[ \Re \left( \langle A_1 \left( \begin{array}{c} u \\ v \end{array} \right) - A_1 \left( \begin{array}{c} \tilde{u} \\ \tilde{v} \end{array} \right) , \left( \begin{array}{c} u \\ v \end{array} \right) - \left( \begin{array}{c} \tilde{u} \\ \tilde{v} \end{array} \right) \rangle \right) \right) \leq 0, \]
let us check, for all \( \left( \begin{array}{c} u \\ v \end{array} \right) \in H \) (= \( H^1_0(0,1) \times L^2(0,1) \)):

\[ \Re \left( \langle A_1 \left( \begin{array}{c} u \\ v \end{array} \right) , \left( \begin{array}{c} u \\ v \end{array} \right) \rangle \right) \right) \leq 0. \]

To do that, using the definition of \( A_1 \), and of the scalar product in \( H^1_0(0,1) \times L^2(0,1) \), it is equal to:

\[
\int_0^1 v'(x)u_x(x)dx + \int_0^1 (u_{xx}(x) - \text{sat}(a v(x)))v(x)dx,
\]

\[
= \int_0^1 v'(x)u_x(x)dx + \int_0^1 u_{xx}(x)v(x)dx - \int_0^1 \text{sat}(a v(x))v(x)dx
\]

\[
= [u_x(x)v(x)]_{x=0}^{x=1} - \int_0^1 \text{sat}(a v(x))v(x)dx \leq 0
\]
due to the boundary and since \( a \geq 0 \).
Second item asks to deal with a nonlinear ODE. Let \((u, v) \in H\) we have to find \((\tilde{u}, \tilde{v}) \in D(A_1)\) such that
\[
(I - \lambda A_1) \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}
\]
that is
\[
\begin{cases}
\tilde{u} - \lambda \tilde{v} = u , \\
\tilde{v} - \lambda (\tilde{u}_{xx} - \text{sat}(a \tilde{v})) = v ,
\end{cases}
\]
In particular, we have to find \(\tilde{u}\) such that
\[
\tilde{u}_{xx} - \frac{1}{\lambda^2} \tilde{u} - \text{sat}\left(\frac{a}{\lambda} (\tilde{u} - u)\right) = -\frac{1}{\lambda} v - \frac{1}{\lambda^2} u
\]
\[
\tilde{u}(0) = \tilde{u}(1) = 0
\]
holds.
Nonhomogeneous nonlinear ODE with two boundary conditions
Lemma

If $a$ is nonnegative and $\lambda$ is positive, then there exists $\tilde{u}$ solution to

$$
\tilde{u}_{xx} - \frac{1}{\lambda^2} \tilde{u} - \text{sat}\left(\frac{a}{\lambda}(\tilde{u} - u)\right) = -\frac{1}{\lambda} v - \frac{1}{\lambda^2} u
$$

$$
\tilde{u}(0) = \tilde{u}(1) = 0
$$

(9)

To prove this lemma, let us introduce the following map

$$
\mathcal{T} : L^2(0, 1) \rightarrow L^2(0, 1),
$$

$$
y \mapsto z = \mathcal{T}(y),
$$

where $z = \mathcal{T}(y)$ is the unique solution to

$$
z_{xx} - \frac{1}{\lambda^2} z = -\frac{1}{\lambda} v - \frac{1}{\lambda^2} u + \text{sat}\left(\frac{a}{\lambda}(y - u)\right),
$$

$$
z(0) = z(1) = 0.
$$

Prove that $\mathcal{T}$ is well defined and apply the Schauder fixed-point theorem (see e.g., [Coron; 2007]), to deduce that there exists $y$ such that $\mathcal{T}(y) = y$

$$
\tilde{u} = y \text{ solves (9)}
$$
Lemma

If \( a \) is nonnegative and \( \lambda \) is positive, then there exists \( \tilde{u} \) solution to

\[
\tilde{u}_{xx} - \frac{1}{\lambda^2} \tilde{u} - \text{sat}(\frac{a}{\lambda}(\tilde{u} - u)) = -\frac{1}{\lambda} v - \frac{1}{\lambda^2} u \\
\tilde{u}(0) = \tilde{u}(1) = 0
\] (9)

To prove this lemma, let us introduce the following map

\[ \mathcal{T} : L^2(0, 1) \rightarrow L^2(0, 1), \]
\[ y \mapsto z = \mathcal{T}(y), \]

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\[
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Prove that \( \mathcal{T} \) is well defined and apply the Schauder fixed-point theorem (see e.g., [Coron; 2007]), to deduce that there exists \( y \) such that \( \mathcal{T}(y) = y \)

\[ \tilde{u} = y \text{ solves (9)} \]
Theorem 2

∀a > 0, for all \((z^0, z^1)\) in 
\((H^2(0, 1) \cap H^1_0(0, 1)) \times H^1_0(0, 1)\), the solution to (8) with the boundary conditions (4) and the initial condition (5) satisfies the following stability property, ∀t ≥ 0,

\[
\|z(\cdot, t)\|_{H^1_0(0,1)} + \|z_t(\cdot, t)\|_{L^2(0,1)} \leq \|z^0\|_{H^1_0(0,1)} + \|z^1\|_{L^2(0,1)},
\]

together with the attractivity property

\[
\|z(\cdot, t)\|_{H^1_0(0,1)} + \|z_t(\cdot, t)\|_{L^2(0,1)} \to 0, \quad \text{as} \ t \to \infty.
\]
Due to Theorem 1, the formal computation

$$
\dot{E} = - \int_0^1 z_t \text{sat}(az_t) \, dx
$$

makes sense. This is only a weak Lyapunov function $\dot{E} \leq 0$
(the state is $(z, z_t)$, and there is no $-z^2$).

To be able to apply LaSalle's Invariance Principle, we have to check that the trajectories are precompact
(see e.g. [Dafermos, Slemrod; 1973]).

It comes from:

**Lemma**

The canonical embedding from $D(A_1)$, equipped with the graph norm, into $H^1_0(0, 1) \times L^2(0, 1)$ is compact.
Sketch of the proof of 

The canonical embedding from $D(A_1)$, equipped with the graph norm, into $H_0^1(0,1) \times L^2(0,1)$ is compact.

Consider a sequence $\left( \begin{array}{c} u_n \\ v_n \end{array} \right)_{n \in \mathbb{N}}$ in $D(A_1)$, which is bounded with the graph norm, that is $\exists M > 0$, $\forall n \in \mathbb{N}$,

$$\left\| \left( \begin{array}{c} u_n \\ v_n \end{array} \right) \right\|_{D(A_1)}^2 := \left\| \left( \begin{array}{c} u_n \\ v_n \end{array} \right) \right\|^2 + \left\| A_1 \left( \begin{array}{c} u_n \\ v_n \end{array} \right) \right\|^2,$$

$$= \int_0^1 (|u_n'|^2 + |v_n|^2 + |v_n'|^2$$

$$+ |u_n''' - \text{asat}(v_n)|^2) \, dx < M$$

From that, we deduce that $\int_0^1 (|v_n|^2 + |v_n'|^2) \, dx$ and $\int_0^1 (|u_n'|^2 + |u_n'''|^2) \, dx$ are bounded.

Thus there exists a subsequence which converges in $H_0^1(0,1) \times L^2(0,1)$. 

□
Using the dissipativity of $A_1$, and previous lemma the trajectory
\[
\begin{pmatrix}
    z(., t) \\
    z_t(., t)
\end{pmatrix}
\]
is precompact in $H^1_0(0, 1) \times L^2(0, 1)$.

Moreover the $\omega$-limit set $\omega \left[ \begin{pmatrix}
    z(., 0) \\
    z_t(., 0)
\end{pmatrix} \right] \subset D(A_1)$, is not empty
and invariant with respect to the nonlinear semigroup $T(t)$ (see
[Slemrod; 1989]).

We now use LaSalle’s invariance principle to show that

\[
\omega \left[ \begin{pmatrix}
    z(., 0) \\
    z_t(., 0)
\end{pmatrix} \right] = \{0\}.
\]

Therefore the convergence property holds. \qed
Remark: Boundary control

1D wave equation with a boundary control.

Dynamics: $\forall x \in (0, 1), t \geq 0,$

$$z_{tt}(x, t) = z_{xx}(x, t),$$

Boundary conditions: $\forall t \geq 0,$

$$z(0, t) = 0,$$
$$z_x(1, t) = -\text{sat}(b z_t(1, t)),$$

In the same work, stability proof using the sector condition + strict Lyapunov function.
For the wave equation+ saturated in-domain control, a non-strict Lyapunov function has been computed.

- Thus a priori no robustness margin.
  What happens in presence of noise?
- For linear PDE, we have exponential convergence (see Proposition on Slide 10).
  Do we have exp. stability for the nonlinear PDE?

Rewrite the wave equation as a abstract control system:

\[
\begin{cases}
\frac{dz}{dt} = Az + Bu, \\
z(0) = z_0.
\end{cases}
\]

There exists a self-adjoint and pos. def. \( P \in \mathcal{L}(H) \) s.t.

\[
\langle (A - BB^*)z, Pz, \rangle_H + \langle Pz, (A - BB^*)z \rangle_H \leq -\|z\|_H^2, \quad \forall z \in D(A)
\]
Consider the $L^2$ saturated case

\[
\begin{align*}
\frac{dz}{dt} &= Az - B\text{sat}_2(B^*z), \\
 z(0) &= z_0,
\end{align*}
\]  

(11)
In presence of saturated input and disturbances

Consider the $L^2$ saturated case + disturbance

\begin{equation}
\begin{aligned}
\frac{dz}{dt} &= Az - B\text{sat}_2(B^*z + d), \\
z(0) &= z_0,
\end{aligned}
\end{equation}

(11)

where $d : (0, \infty) \rightarrow L^2(0, 1)$ is a disturbance.
Consider the $L^2$ saturated case + disturbance

\[
\begin{aligned}
\frac{dz}{dt} &= Az - B_{\text{sat}_2}(B^*z + d), \\
z(0) &= z_0,
\end{aligned}
\]

where $d : (0, \infty) \to L^2(0,1)$ is a disturbance.

Recall the $L^2$ saturation: Given $u : [0, 1] \to \mathbb{R}$, $\text{sat}_2(\sigma)$ is the function defined by $\text{sat}_2(\sigma) = \begin{cases} 
\sigma & \text{if } \|\sigma\|_{L^2(0,1)} < 1 \\
\frac{\sigma}{\|\sigma\|_{L^2(0,1)}} & \text{else}
\end{cases}$
Consider the $L^2$ saturated case + disturbance

$$\begin{cases} \frac{dz}{dt} = Az - B \text{sat}_2(B^*z + d), \\ z(0) = z_0, \end{cases} \quad (11)$$

where $d : (0, \infty) \rightarrow L^2(0, 1)$ is a disturbance.

Recall the $L^2$ saturation: Given $u : [0, 1] \rightarrow \mathbb{R}$, $\text{sat}_2(\sigma)$ is the function defined by $\text{sat}_2(\sigma) = \begin{cases} \sigma & \text{if } \|\sigma\|_{L^2(0,1)} < 1 \\ \frac{\sigma}{\|\sigma\|_{L^2(0,1)}} & \text{else} \end{cases}$

What can be said about the exp. stability when $d = 0$ and about the robustness in presence of $d$?
ISS notion

**Input-to-State Stability (ISS) definition**

A positive definite function $V : H \rightarrow \mathbb{R}_{\geq 0}$ is said to be an ISS-Lyapunov function with respect to $d$ if $\exists$ two class $\mathcal{K}_\infty$ functions $\alpha$ and $\rho$ such that, for any solution to (11)

$$\frac{d}{dt} V(z) \leq -\alpha(\|z\|_H) + \rho(\|d\|_{L^2(0,1)}).$$

**Remark:** Of course ISS Lyapunov function $+ \exists$ two functions $\underline{\alpha}$ and $\overline{\alpha}$ of class$^1 \mathcal{K}$ such that

$$\underline{\alpha}(\|z\|_H) \leq V(z) \leq \overline{\alpha}(\|z\|_H), \forall z \in H$$

$\Rightarrow$ the origin of (11) with $d = 0$ is globally asymptotically stable.

---

$^1\alpha : [0, \infty) \rightarrow [0, \infty)$ is of class $\mathcal{K}$ if it is continuous, zero at zero and increasing. It is of class $\mathcal{K}_\infty$ if it is moreover unbounded.
**Theorem 3** [Marx, Chitour, CP; to appear]

Suppose that Assumption 1 holds and let $P \in \mathcal{L}(H)$ be a self-adjoint and positive operator satisfying (10). Then, there exists $M$ such that

$$V(z) := \langle Pz, z \rangle_H + M\|z\|_H^3$$

(12)

is an ISS-Lyapunov function for (11).

The proof follows the finite-dimensional case considered in [Liu, Chitour, and Sontag; 1996].
Sketch of the proof

Let us consider the following candidate Lyapunov function

\[ V(z) := \langle Pz, z \rangle_H + M\|z\|^3_H \]

Along the strong solutions to (11), with \( \tilde{A} = A - BB^* \)

\[
\frac{d}{dt} \langle Pz, z \rangle_H = \langle Pz, Az \rangle_H + \langle PAz, z \rangle_H \\
+ 2\langle PB(\text{sat}_2(B^*z) - \text{sat}_2(B^*z + d)), z \rangle_H
\]
Let us consider the following candidate Lyapunov function

$$V(z) := \langle Pz, z \rangle_H + M\|z\|_H^3$$

Along the strong solutions to (11), with $\tilde{A} = A - BB^*$

$$\frac{d}{dt} \langle Pz, z \rangle_H = \langle Pz, \tilde{A}z \rangle_H + \langle P\tilde{A}z, z \rangle_H$$

$$+ 2\langle PB(B^*z - \text{sat}_2(B^*z), z \rangle_H$$

$$+ 2\langle PB(\text{sat}_2(B^*z) - \text{sat}_2(B^*z + d)), z \rangle_H$$
Let us consider the following candidate Lyapunov function
\[
V(z) := \langle Pz, z \rangle_H + M\|z\|^3_H
\]
Along the strong solutions to (11), with \( \tilde{A} = A - BB^* \)
\[
\frac{d}{dt} \langle Pz, z \rangle_H = \langle Pz, \tilde{A}z \rangle_H + \langle P\tilde{A}z, z \rangle_H
\]
\[
+ 2\langlePB(B^*z - \text{sat}_U(B^*z)), z \rangle_H
\]
\[
+ 2\langlePB(\text{sat}_2(B^*z) - \text{sat}_2(B^*z + d)), z \rangle_H
\]
\[
\leq -\|z\|^2_H + 2\|B^*z\|_{L^2(0,1)}\|P\|_{L(H)}\|B^*z - \text{sat}_2(B^*z)\|_{L^2(0,1)}
\]
\[
+ 2\langle\text{sat}_2(B^*z) - \text{sat}_2(B^*z + d), B^*Pz \rangle_{L^2(0,1)},
\]
Sketch of the proof

Let us consider the following candidate Lyapunov function

$$V(z) := \langle Pz, z \rangle_H + M\|z\|_H^3$$

Along the strong solutions to (11), with $\tilde{A} = A - BB^*$

$$\frac{d}{dt}\langle Pz, z \rangle_H = \langle Pz, \tilde{A}z \rangle_H + \langle P\tilde{A}z, z \rangle_H$$

$$+ 2\langle PB(B^*z - \text{sat}_2(B^*z)), z \rangle_H$$

$$+ 2\langle PB(\text{sat}_2(B^*z) - \text{sat}_2(B^*z + d)), z \rangle_H$$

$$\leq -\|z\|_H^2 + 2\|B^*z\|_2\|P\|_{\mathcal{L}(H)}\|B^*z - \text{sat}_{L^2(0,1)}(B^*z)\|_{L^2(0,1)}$$

$$+ 2\langle \text{sat}_2(B^*z) - \text{sat}_2(B^*z + d), B^*Pz \rangle_{L^2(0,1)},$$

$$\leq -\|z\|_H^2 + 2\|B^*z\|_{L^2(0,1)}\|P\|_{\mathcal{L}(H)}\|B^*z - \text{sat}_2(B^*z)\|_{L^2(0,1)}$$

$$+ 2k\|d\|_{L^2(0,1)}\|B^*\|_{\mathcal{L}(H,L^2(0,1))}\|P\|_{\mathcal{L}(H)}\|z\|_H,$$

using $\text{sat}_2$ Lipchitz, Cauchy-Schwarz inequality and the fact that $B^*$ is bounded.
Moreover using $\|d\|_{L^2(0,1)} \|z\|_H \leq \varepsilon \|d\|_{L^2(0,1)}^2 + \frac{1}{\varepsilon} \|z\|_H^2$ and $\|B^*z - \text{sat}_2(B^*z)\|_{L^2(0,1)} \leq \langle \text{sat}_2(B^*z), B^*z \rangle_{L^2(0,1)}$, we get

$$
\frac{d}{dt} \langle Pz, z \rangle_H \leq - \left( 1 - \frac{\|B^*\|_{L(H,L^2(0,1))}^2 \|P\|_{L(H)}^2}{\varepsilon_1} \right) \|z\|_H^2 
+ 2 \|B^*\|_{L(H,L^2(0,1))} \|P\|_{L(H)} \|z\|_H \langle \text{sat}_2(B^*z), B^*z \rangle_{L^2(0,1)} 
+ k^2 \varepsilon_1 \|d\|_{L^2(0,1)}^2
$$

where $\varepsilon_1$ is a positive value that will be selected later.

Thus

$$
\frac{d}{dt} W(z) \leq \text{good term} + \text{bad term} + d^2
$$
Secondly, using the dissipativity of the operator $A_{sat}$,

$$\langle sat_2(B^*z) - sat_2(B^*z + d), B^*z \rangle_{L^2(0,1)} \leq C_0 \| d \|_{L^2(0,1)},$$

and

$$\| z \|_{H} \| d \|_{L^2(0,1)} \leq \frac{1}{\varepsilon_2} \| z \|_{H}^2 + \varepsilon_2 \| d \|_{L^2(0,1)}^2,$$

one has

$$\frac{2M}{3} \frac{d}{dt} \| z \|_{H}^3 = M \| z \| ((\langle Az, z \rangle_{H} + \langle z, Az \rangle_{H})$$

$$- 2M \| z \|_{H} \langle Bsat_2(B^*z + d), z \rangle_{H}$$

$$\leq - 2M \| z \|_{H} \langle \langle sat_2(B^*z), B^*z \rangle_{L^2(0,1)}$$

$$+ \langle sat_2(B^*z) - sat_2(B^*z + d), B^*z \rangle_{L^2(0,1)} \rangle_{H}$$

$$\leq - 2M \| z \|_{H} \langle sat_2(B^*z), B^*z \rangle_{L^2(0,1)}$$

$$+ 2MC_0 \| z \|_{H} \| d \|_{L^2(0,1)}$$

$$\leq - 2M \| z \|_{H} \langle sat_2(B^*z), B^*z \rangle_{L^2(0,1)}$$

$$+ \frac{2MC_0}{\varepsilon_2} \| z \|_{H}^2 + 2MC_0 \varepsilon_2 \| d \|_{L^2(0,1)}^2,$$

where $\varepsilon_2$ is a positive value that has to be selected. For an appropriate choice of $M$, $\varepsilon_1$ and $\varepsilon_2$ we deduce the result.

\[\square\]
Secondly, using the dissipativity of the operator $A_{\text{sat}}$, 
\[
\langle \text{sat}_2(B^*z) - \text{sat}_2(B^*z + d), B^*z \rangle_{L^2(0,1)} \leq C_0 \|d\|_{L^2(0,1)},
\]
and 
\[
\|z\|_H \|d\|_{L^2(0,1)} \leq \frac{1}{\varepsilon_2} \|z\|_H^2 + \varepsilon_2 \|d\|_{L^2(0,1)}^2,
\]
one has
\[
\frac{2M}{3} \frac{d}{dt} \|z\|_H^3 = M \|z\| \left( \langle Az, z \rangle_H + \langle z, Az \rangle_H \right) 
- 2M \|z\|_H \langle B \text{sat}_2(B^*z + d), z \rangle_H 
\leq -2M \|z\|_H \langle \text{sat}_2(B^*z), B^*z \rangle_{L^2(0,1)} 
+ \langle \text{sat}_2(B^*z) - \text{sat}_2(B^*z + d), B^*z \rangle_{L^2(0,1)} \rangle 
\leq -2M \|z\|_H \langle \text{sat}_2(B^*z), B^*z \rangle_{L^2(0,1)} 
+ 2MC_0 \|z\|_H \|d\|_{L^2(0,1)} 
\leq -2M \|z\|_H \langle \text{sat}_2(B^*z), B^*z \rangle_{L^2(0,1)} 
+ \frac{2MC_0}{\varepsilon_2} \|z\|_H^2 + 2MC_0 \varepsilon_2 \|d\|_{L^2(0,1)}^2,
\]
where $\varepsilon_2$ is a positive value that has to be selected. For an appropriate choice of $M$, $\varepsilon_1$ and $\varepsilon_2$ we deduce the result.
Secondly, using the dissipativity of the operator $A_{\text{sat}}$, 
\[
\langle \text{sat}_2(B^*z) - \text{sat}_2(B^*z + d), B^*z \rangle_{L^2(0,1)} \leq C_0 \| d \|_{L^2(0,1)}, \quad \text{and}
\| z \|_H \| d \|_{L^2(0,1)} \leq \frac{1}{\varepsilon_2} \| z \|_H^2 + \varepsilon_2 \| d \|_{L^2(0,1)}^2,
\]
one has
\[
\frac{2M}{3} \frac{d}{dt} \| z \|_H^3 = M \| z \| (\langle Az, z \rangle_H + \langle z, Az \rangle_H)
- 2M \| z \|_H \langle B \text{sat}_2(B^*z + d), z \rangle_H
\leq - 2M \| z \|_H (\langle \text{sat}_2(B^*z), B^*z \rangle_{L^2(0,1)}
+ \langle \text{sat}_2(B^*z) - \text{sat}_2(B^*z + d), B^*z \rangle_{L^2(0,1)})
\leq - 2M \| z \|_H \langle \text{sat}_2(B^*z), B^*z \rangle_{L^2(0,1)}
+ 2MC_0 \| z \|_H \| d \|_{L^2(0,1)}
\leq -2M \| z \|_H \langle \text{sat}_2(B^*z), B^*z \rangle_{L^2(0,1)}
+ \frac{2MC_0}{\varepsilon_2} \| z \|_H^2 + 2MC_0 \varepsilon_2 \| d \|_{L^2(0,1)}^2,
\]
where $\varepsilon_2$ is a positive value that has to be selected. For an appropriate choice of $M$, $\varepsilon_1$ and $\varepsilon_2$ we deduce the result. □
Secondly, using the dissipativity of the operator $A_{\text{sat}}$, 
\[ \langle \text{sat}_2(B^*z) - \text{sat}_2(B^*z + d), B^*z \rangle_{L^2(0,1)} \leq C_0 \| d \|_{L^2(0,1)}, \] and 
\[ \| z \|_H \| d \|_{L^2(0,1)} \leq \frac{1}{\varepsilon_2} \| z \|_H^2 + \varepsilon_2 \| d \|_{L^2(0,1)}^2, \] one has 
\[ \frac{2M}{3} \frac{d}{dt} \| z \|_H^3 = M \| z \| (\langle Az, z \rangle_H + \langle z, Az \rangle_H) \]
\[ - 2M \| z \|_H \langle B \text{sat}_2(B^*z + d), z \rangle_H \]
\[ \leq - 2M \| z \|_H (\langle \text{sat}_2(B^*z), B^*z \rangle_{L^2(0,1)} \]
\[ + \langle \text{sat}_2(B^*z) - \text{sat}_2(B^*z + d), B^*z \rangle_{L^2(0,1)} \]
\[ \leq - 2M \| z \|_H \langle \text{sat}_2(B^*z), B^*z \rangle_{L^2(0,1)} \]
\[ + 2MC_0 \| z \|_H \| d \|_{L^2(0,1)} \]
\[ \leq -2M \| z \|_H \langle \text{sat}_2(B^*z), B^*z \rangle_{L^2(0,1)} \]
\[ + \frac{2MC_0}{\varepsilon_2} \| z \|_H^2 + 2MC_0 \varepsilon_2 \| d \|_{L^2(0,1)}^2, \]
\[ \varepsilon_2 \] is a positive value that has to be selected. For an appropriate choice of $M$, $\varepsilon_1$ and $\varepsilon_2$ we deduce the result. \qed
What happens with $sat$ instead of $sat_2$? What is the speed of convergence of

$$\begin{cases} \frac{d}{dt} z = Az - B \text{sat}(B^*z), \\ z(0) = z_0, \end{cases} \quad (13)$$
Theorem

Hence, the origin of (13) is semi-globally exponentially stable in $D(A)$, that is for any positive $r$ and any $z_0$ in $D(A)$ satisfying $\|z_0\|_{D(A)} \leq r$, there exist two positive constants $\mu := \mu(r)$ and $K := K(r)$ such that

$$\|W_\sigma(t)z_0\|_H \leq Ke^{-\mu t}\|z_0\|_H, \quad \forall t \geq 0.$$  \hspace{1cm} (14)

Remarks • on Korteweg-de Vries equation: [Rosier, Zhang; 2006] and [Marx, Cerpa, CP, Andrieu; 2017]. We may deduce a global asymptotic stability (but without any estimation of the convergence speed). • In our work the monotonicity is crucial and also only 1D. See [Martinez, Vancostenoble; 2000] for $N \leq 2$. See also last part of this presentation for $N = 1$. 

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Remarks • on Korteweg-de Vries equation: [Rosier, Zhang; 2006] and [Marx, Cerpa, CP, Andrieu; 2017]. We may deduce a global asymptotic stability (but without any estimation of the convergence speed).

• In our work the monotonicity is crucial and also only 1D

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Sketch of the proof

Let $\tilde{V}(z)$ be the Lyapunov function candidate defined by

$$z \in D(A) \mapsto \tilde{V}(z) := \langle Pz, z \rangle_H + \tilde{M}\|z\|^2_H,$$

where $\tilde{M} > 0$ will be selected later. As before, using the dissipativity of the operator $A - B^*B$, one has

$$\frac{d}{dt} \tilde{M}\|z\|^2_H \leq -2\tilde{M}\langle B^*z, \text{sat}(B^*z) \rangle_{L^2(0,1)}. \tag{15}$$

and

$$\frac{d}{dt} \langle Pz, z \rangle_H \leq -\|z\|^2_H + 2\langle B^*Pz, B^*z - \text{sat}(B^*z) \rangle_{L^2(0,1)}.$$

The term $2\langle B^*Pz, B^*z - \text{sat}(B^*z) \rangle_{L^2(0,1)}$ is ”controlled” differently.
Consider $r > 0$ and a strong solution for (13), whose initial condition $z_0 \in D(A)$ is such that

$$\|z_0\|_{D(A)} \leq r$$

First note that, from the dissipativity, it implies $\|z(t)\|_{D(A)} \leq r$ for all $t \geq 0$.

$$\left| \langle B^*Pz, B^*z - \text{sat}(B^*z) \rangle_{L^2(0,1)} \right| \\
\leq \|B^*Pz\|_{L^\infty(0,1)} \|B^*z - \text{sat}(B^*z)\|_{L^1(0,1)} \\
\leq C \|Pz\|_{D(A)} \|B^*z - \text{sat}(B^*z)\|_{L^1(0,1)} \\
\leq C' \|P\|_{\mathcal{L}(D(A))} \|z\|_{D(A)} \langle \text{sat}(B^*z), B^*z \rangle_{L^2(0,1)}$$

Therefore

$$\frac{d\tilde{V}}{dt} \leq -\|z\|_H^2 - 2(\tilde{M} - C' \|P\|_{\mathcal{L}(D(A))}) \|z\|_{D(A)} \langle \text{sat}(B^*z), B^*z \rangle \\
\leq -\|z\|_H^2 - 2(\tilde{M} - C' \|P\|_{\mathcal{L}(D(A))}r) \langle \text{sat}(B^*z), B^*z \rangle \\
\leq -\|z\|_H^2$$

for a suitable $\tilde{M}$. The result follows.
consider again the controlled wave equation:

\[
\begin{cases}
z_{tt} = z_{xx} + u, & (t, x) \in \mathbb{R}_+ \times [0, 1] \\
z(t, 0) = z(t, 1) = 0, & t \in \mathbb{R}_+ \\
z(0, x) = z_0(x), & z_t(0, x) = z_1(x), & x \in [0, 1],
\end{cases}
\]

Nonlinear damping \( \sigma \) law given by the damping

\[ u(t, x) = -\sqrt{a(x)} \sigma(\sqrt{a(x)} z_t(t, x)) \] where \( \forall x \in \omega, a_0 < a(x) \leq a_\infty, a_0 > 0 \).
A function $\sigma$ is a nonmonotone damping if

1. it is locally Lipschitz
2. $\sigma(0) = 0$
3. for all $s \in \mathbb{R}$, $\sigma(s)s > 0$
4. the function $\sigma$ is differentiable at $s = 0$ with $\sigma'(0) = C_1$, where $C_1$ is a positive constant.
Nonmonotone damping

For example: \( \sigma(s) = \text{sat}\left(\frac{1}{4}s - \frac{1}{30}\sin(10s)\right) \)
Regularity issues

Since the function $\sigma$ is (possibly) nonmonotone, then the LaSalle’s Invariance Principle does not apply!

Moreover, the classical functional setting

$$H = H^1_0(0,1) \times L^2(0,1),$$

is not sufficient to ensure a $L^\infty$ regularity for the state $z_t$.

Solution (inspired by [Haraux; 2009])

Our solution consists in using the functional setting

$$H_p := (W^{1,p}(0,1) \cap H^1_0(0,1)) \times L^p(0,1),$$

where $p \in [1, \infty]$. 
Main results

\[
\begin{aligned}
z_{tt} & = z_{xx} - \sqrt{a(x)} \sigma(\sqrt{a(x)} z_t), \quad (t, x) \in \mathbb{R}_+ \times [0, 1] \\
z(t, 0) & = z(t, 1) = 0, \quad t \in \mathbb{R}_+ \\
z(0, x) & = z_0(x), \quad z_t(0, x) = z_1(x), \quad x \in [0, 1]. \\
\end{aligned}
\]

(System) (Sys)

Theorem [Chitour, Marx, CP; under submission] (well-posedness)

\[
\forall \text{ initial condition } (z_0, z_1) \in H_\infty, \exists! \text{ solution } (z, z_t) \in L^\infty(\mathbb{R}_+; W^{1,\infty}(0, 1)) \times W^{1,\infty}(\mathbb{R}_+; L^\infty(0, 1)) \text{ to (Sys).}
\]

Moreover, one has

\[
\| (z, z_t) \|_{H_\infty(0,1)} \leq 2 \max (\| z'_0 \|_{L^\infty(0,1)}, \| z_1 \|_{L^\infty(0,1)})
\]
Main results

\[
\begin{aligned}
\left\{
\begin{array}{l}
z_{tt} = z_{xx} - \sqrt{a(x)}\sigma(\sqrt{a(x)}z_t), \ (t, x) \in \mathbb{R}_+ \times [0, 1] \\
z(t, 0) = z(t, 1) = 0, \ t \in \mathbb{R}_+ \\
z(0, x) = z_0(x), \ z_t(0, x) = z_1(x), \ x \in [0, 1].
\end{array}
\right.
\end{aligned}
\]  
(Sys)

Theorem [Chitour, Marx, CP; under submission] (convergence)

Given \( r > 0 \). Consider initial conditions in \( H_\infty \) satisfying

\[
\| (z_0, z_1) \|_{H_\infty} \leq r.
\]

Then, \( \forall p \in [2, \infty), \exists K := K(r) \) and \( \mu := \mu(r) \) such that

\[
\| (z, z_t) \|_{H_p} \leq Ke^{-\mu t}\| (z_0, z_1) \|_{H_p}, \ \forall t \geq 0.
\]
Well-posedness proof (1)

- **Fixed-point theorem** ⇒ existence and uniqueness in $[0, T]$.
- The estimate is proved thanks to the following result

**Theorem [Haraux; 2009]**

Let us consider initial condition in $H_\infty$. Let us introduce the following functional

$$
\phi(z, z_t) = \int_0^1 [F(z - z_t) + F(z + z_t)]dx,
$$

where $F$ is any even and convex function. Then, the time derivative of $\phi$ along the trajectories of (Sys) satisfies

$$
\frac{d}{dt}\phi(z, z_t) \leq 0.
$$
Due to the latter theorem, one has \(\phi(z, z_t) \leq \phi(z_0, z_1)\), for all \(t \geq 0\). Then, the result follows by setting

\[
F(s) := \text{Pos}(|s| - 2 \max(\|z'_0\|_{L^\infty(0,1)}, \|z_1\|_{L^\infty(0,1)})
\]

where

\[
\text{Pos}(s) := \begin{cases} 
s & \text{if } s > 0, \\
0 & \text{if } s \leq 0. 
\end{cases}
\]

This implies that \(\phi(z, z_t) = 0\) and then, for all \(t \geq 0\)

\[
\| (z, z_t) \|_{H^\infty(0,1)} \leq 2 \max (\|z'_0\|_{L^\infty(0,1)}, \|z_1\|_{L^\infty(0,1)}).
\]

**Question**

What about the asymptotic stability?
Consider (Sys), with initial conditions in $H_\infty$. Thanks to this regularity:

1. Prove the result in $H_2 = H_0^1(0, 1) \times L^2(0, 1)$
2. Deduce the result in $H_p$ by an interpolation theorem (Riesz-Thaurin theorem), with

$$H_p = (W^{1,p}(0, 1) \cap H_0^1(0, 1)) \times L^p(0, 1)$$

**Strategy**

Transforming the nonlinear time-invariant system as a trajectory of a linear time-variant system.
A detour via linear time-variant systems

System (Sys) can be seen as a trajectory of a linear time-variant system (LTV).

\[
\begin{aligned}
\begin{cases}
  z_{tt} = z_{xx} - a(x)d(t, x)z_t, & (t, x) \in \mathbb{R}_+ \times [0, 1], \\
  z(t, 0) = z(t, 1) = 0, & t \in \mathbb{R}_+, \\
  z(0, x) = z_0(x), & z_t(0, x) = z_1(x), \\
  & x \in [0, 1],
\end{cases}
\end{aligned}
\]  

(LTV-wave)

where

\[
d(t, x) = \begin{cases}
  \frac{\sigma(\sqrt{a(x)}z_t)}{\sqrt{a(x)}z_t}, & \sqrt{a(x)}z_t \neq 0, \\
  C_1, & \sqrt{a(x)}z_t = 0,
\end{cases}
\]

where \(C_1 = \sigma'(0)\).
Let us recall that $H_2 = H^1_0(0,1) \times L^2(0,1)$ and let us introduce $U = L^2(0,1)$. Consider the abstract system

$$\begin{cases}
  \frac{d}{dt} y = Ay - d(t)BB^* y := A_d(t)y, \\
y(\tau) = y_\tau, \tau \geq 0,
\end{cases} \quad \text{(Abstract)}$$

with $y = [z \ z_t]^\top$, $A : D(A) \subset H_2 \to H_2$ defined as

$$A = \begin{bmatrix} 0 & l_{H_2} \\ \partial_{xx} & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & \sqrt{a(x)} l_{H_2} \end{bmatrix}^\top,$$

with $D(A) = (H^2(0,1) \cap H^1_0(0,1)) \times H^1_0(0,1)$.

**Trajectories**

(Sys) and (Abstract) share one trajectory, i.e. when $\tau = 0$. 
Proposition (for convergence result)

Suppose that there exist $d_0, d_1 > 0$ such that

$$d_0 \leq d(t) \leq d_1.$$ 

Then, if

$$\frac{d}{dt} y = Ay - d_0 BB^* y := A_{d_0} y,$$

$$y(0) = y_0,$$

is exponentially stable, the trajectory of (Abstract) with $\tau = 0$ converges to 0.
Lyapunov proof of this proposition

Exponential stability $\Rightarrow \exists \hat{P} \in \mathcal{L}(H_2)$ and $C > 0$ such that

$$\langle \hat{P}y, A_{d_0} \rangle_{H_2} + \langle \hat{P}A_{d_0}y, y \rangle_{H_2} \leq -C\|y\|_{H_2}^2$$

Time derivative of the Lyapunov functional

$$\hat{V}(y) := \langle \hat{P}y, y \rangle_{H_2} + \hat{M}\|y\|_{H_2}^2$$

along the trajectories of (Abstract) with $\hat{M} = \frac{2(d_1-d_0)\|\hat{P}\|_{\mathcal{L}(H_2)}}{d_0\|B\|_{\mathcal{L}(H_2, U)}}$,

$$\frac{d\hat{V}}{dt}(y) \leq -C\|y\|_{H_2}^2$$

Then,

$$\|y\|_{H_2}^2 \leq \frac{\|\hat{P}\|_{\mathcal{L}(H_2)}}{\hat{M}} + \hat{M} \exp \left(-\frac{C}{\|\hat{P}\|_{\mathcal{L}(H_2)} + \hat{M}}t\right) \|y_0\|_{H_2}^2, \forall t \geq 0$$
Back to the proof of convergence result

Recall that

\[
    d(t, x) = \begin{cases} 
    \frac{\sigma(\sqrt{a(x)}z_t)}{\sqrt{a(x)}z_t}, & \sqrt{a(x)}z_t \neq 0, \\
    C_1, & \sqrt{a(x)}z_t = 0,
    \end{cases}
\]

and that

\[
    \| (z, z_t) \|_{H_\infty(0,1)} \leq 2 \max \left( \| z_0 \|_{L_\infty(0,1)}, \| z_1 \|_{L_\infty(0,1)} \right) \leq 2r,
\]

then

\[
    d_0 := \min_{\xi \in [-2\sqrt{a_\infty}r, 2\sqrt{a_\infty}r]} \frac{\sigma(\xi)}{\xi} \leq d(t, x) \leq \max_{\xi \in [-2\sqrt{a_\infty}r, 2\sqrt{a_\infty}r]} \frac{\sigma(\xi)}{\xi} := d_1.
\]

Then, one can prove easily that

\[
    \| (z, z_t) \|_{H_2} \leq K(r)e^{-\mu(r)t}\| (z_0, z_1) \|_{H_2}
\]

which is the result.
### Results

1. Asymptotic stability in $H_p$ for non-monotone damping
2. Semi-global exponential stability in $H$ for monotone damping
3. Instead of wave equations, abstract operator theories could be developed

### Further research lines

1. What about quasilinear hyperbolic systems

   \[
   \begin{aligned}
   z_t + \Lambda(z)z_x &= 0 \\
   z(t, 0) &= Hz(t, 1) + Bu(t)
   \end{aligned}
   \]

   See [Coron, Ervedoza, Ghoshal, Glass, Perrollaz; 17], and the current work of M. Dus for BV solutions.

2. $N$-dimensional wave equations?
   $N \leq 2$ in [Martinez, Vancostenoble; 2000]
1D wave equation with a boundary control.

Dynamics:

\[ z_{tt}(x, t) = z_{xx}(x, t), \quad \forall x \in (0, 1), \quad t \geq 0, \]  

(16)

Boundary conditions, \( \forall t \geq 0 \),

\[ z(0, t) = 0, \quad z_x(1, t) = g(t), \]  

(17)

and with the same initial condition, \( \forall x \in (0, 1) \),

\[ z(x, 0) = z^0(x), \quad z_t(x, 0) = z^1(x). \]  

(18)
When closing the loop with a linear boundary control

Let us define the linear control by

\[ g(t) = -bz_t(1, t), \ x \in (0, 1), \ \forall \ t \geq 0 \]  

(19)

and consider

\[ E_2 = \frac{1}{2} \int (e^{\mu x}(z_t + z_x)^2 \, dx + \int (e^{-\mu x}(z_t - z_x)^2 \, dx, \]

Formal computation. Along the solutions to (16), (17) and (19):

\[ \dot{E}_2 = -\mu E_2 + \frac{1}{2} \left( e^{\mu}(1 - b)^2 - e^{-\mu}(1 + b)^2 \right) z_t^2(1, t) \]

Assuming \( b > 0 \) and letting \( \mu > 0 \) such that \( e^{\mu}(1 - b)^2 \leq e^{-\mu}(1 + b)^2 \), it holds \( \dot{E}_2 \leq -\mu E_2 \) and thus \( E_2 \) is a strict Lyapunov function and thus (16)-(19) is exponentially stable.
When closing the loop with a linear boundary control

Let us define the linear control by

\[ g(t) = -bz_t(1, t), \ x \in (0, 1), \ \forall t \geq 0 \]  \hspace{1cm} (19)

and consider

\[ E_2 = \frac{1}{2} \int (e^{\mu x}(z_t + z_x)^2 \, dx + \int (e^{-\mu x}(z_t - z_x)^2 \, dx, \]

**Formal computation.** Along the solutions to (16), (17) and (19):

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it holds \( \dot{E}_2 \leq -\mu E_2 \) and thus \( E_2 \) is a strict Lyapunov function and thus (16)-(19) is exponentially stable.
When closing the loop with a saturating control

Let us consider now the nonlinear control
\[ g(t) = -\text{sat}(b z_t(1, t)), \forall t \geq 0. \]
The boundary conditions become:
\[ z(0, t) = 0, \quad z_x(1, t) = -\text{sat}(b z_t(1, t)) \]. \quad (20)

Theorem (stability with boundary control)

\[ \forall b > 0, \text{ for all } (z^0, z^1) \text{ in } \{ (u, v), (u, v) \in H^2(0, 1) \times H^1_0(0, 1), \ u_x(1) + \text{sat}(b v(1)) = 0, \ u(0) = 0 \}, \text{ the solution to (16) with the boundary conditions (20) and the initial condition (5) satisfies the following stability property, } \forall t \geq 0, \]
\[ \|z(\cdot, t)\|_{H^1_0(0, 1)} + \|z_t(\cdot, t)\|_{L^2(0, 1)} \leq \|z^0\|_{H^1_0(0, 1)} + \|z^1\|_{L^2(0, 1)}, \]

together with the attractivity property
\[ \|z(\cdot, t)\|_{H^1_0(0, 1)} + \|z_t(\cdot, t)\|_{L^2(0, 1)} \rightarrow 0, \text{ as } t \rightarrow \infty. \]
To prove the well-posedness of the Cauchy problem we prove that $A_2$ defined by

$$A_2 \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ u'' \end{pmatrix}$$

with the domain $D(A_2) = \{(u, v), (u, v) \in H^2(0, 1) \times H^1_0(0, 1), u'(1) + \text{sat}(b v(1)) = 0, u(0) = 0\}$ is a semigroup of contraction.

The global stability property comes directly from the dissipativity of $A_2$.

The global attractivity property comes from the following lemma:
To prove the well-posedness of the Cauchy problem we prove that $A_2$ defined by

$$A_2 \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ u'' \end{pmatrix}$$

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The **global stability property** comes directly from the dissipativity of $A_2$.

The **global attractivity property** comes from the following lemma:
Lemma (semi-global exponential stability)

For all $r > 0$, there exists $\mu > 0$ such that, for all initial condition satisfying

$$\|z^{0''}\|_{L^2(0,1)}^2 + \|z^1\|_{H^1_0(0,1)}^2 \leq r^2,$$  \hspace{1cm} (21)

it holds

$$\dot{E}_2 \leq -\mu E_2$$

along the solutions to (16) with the boundary conditions (20).
Sketch of the proof of this lemma

First note that by dissipativity of $A_2$, it holds that

$$
t \mapsto \left\| A_2 \begin{pmatrix} \cdot \cdot \cdot (z(\cdot, t) \\ z_t(\cdot, t) \end{pmatrix} \right\|_H$$

is a non-increasing function. Thus, for all $t \geq 0$,

$$|z_t(1, t)| \leq \left\| A_2 \begin{pmatrix} \cdot \cdot \cdot (z(\cdot, 0) \\ z_t(\cdot, 0) \end{pmatrix} \right\|_H.$$ 

Now for all initial conditions satisfying (21), there exists $c \neq b$ such that, for all $t \geq 0$,

$$(b - c)|z_t(1, t)| \leq 1$$

and thus the following local sector condition holds:

Letting $\sigma = z_t(1, t)$, it holds

$$\left( \text{sat}(b\sigma) - b\sigma \right)\left( \text{sat}(b\sigma) - (b - c)\sigma \right) \leq 0$$
We come back to the Lyapunov function candidate $E_2$. Given $b > 0$, using the previous inequality, we compute

\[
\dot{E}_2 = -\mu E_2 + e^\mu (\sigma - \text{sat}(b\sigma))^2 - e^{-\mu} (\sigma + \text{sat}(b\sigma))^2
\]

\[
\leq -\mu E_2 + \begin{pmatrix} \sigma \\ \text{sat}(b\sigma) \end{pmatrix}^T \begin{pmatrix} e^\mu - e^{-\mu} - b^2(b-c) & -e^\mu - e^{-\mu} + b + b(b-c) \\ -e^\mu - e^{-\mu} + b + b(b-c) & -1 + e^\mu - e^{-\mu} \end{pmatrix} \begin{pmatrix} \sigma \\ \text{sat}(b\sigma) \end{pmatrix}
\]

\[
\leq -\mu E_2
\]

with a suitable choice of constant values $\mu$ and $c$. The semi-global exponential stability follows. □
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\dot{E}_2 = -\mu E_2 + e^{\mu}(\sigma - \text{sat}(b\sigma))^2 - e^{-\mu}(\sigma + \text{sat}(b\sigma))^2
\]

\[
\leq -\mu E_2 + \left( \begin{array}{c} \sigma \\ \text{sat}(b\sigma) \end{array} \right)^T \left( \begin{array}{cc} e^\mu - e^{-\mu} - b^2(b-c) & -e^\mu - e^{-\mu} + b + b(b-c) \\ -e^\mu - e^{-\mu} + b + b(b-c) & -1 + e^\mu - e^{-\mu} \end{array} \right) \left( \begin{array}{c} \sigma \\ \text{sat}(b\sigma) \end{array} \right)
\]

\[
\leq -\mu E_2
\]

with a suitable choice of constant values $\mu$ and $c$. The semi-global exponential stability follows. \[\square\]
We come back to the Lyapunov function candidate $E_2$. Given $b > 0$, using the previous inequality, we compute

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\leq -\mu E_2 + \begin{pmatrix} \sigma \\ \text{sat}(b\sigma) \end{pmatrix}^\top \begin{pmatrix} e^\mu - e^{-\mu} - b^2 (b - c) & -e^\mu - e^{-\mu} + b + b(b - c) \\ -e^\mu - e^{-\mu} + b + b(b - c) & -1 + e^\mu - e^{-\mu} \end{pmatrix} \begin{pmatrix} \sigma \\ \text{sat}(b\sigma) \end{pmatrix} \\
\leq -\mu E_2
\]

with a suitable choice of constant values $\mu$ and $c$. The semi-global exponential stability follows. $\square$

Back to the wave equation with in-domain control