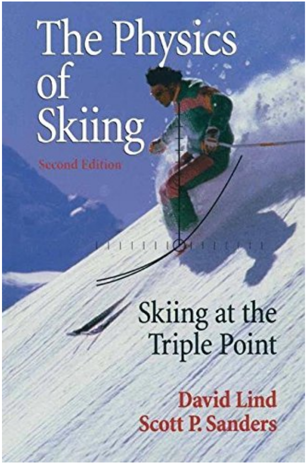
Wave equation and nonlinear damping controls

Christophe PRIEUR
CNRS, Gipsa-lab, Grenoble, France

Control and stabilization issues for PDE
dedicated to Jean-Pierre Raymond



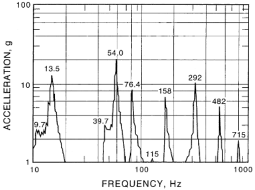


FIGURE 3.5. The frequency distribution of the normal vibration modes of a ski. The ski is clamped at the center to a shaker and driven. An output accelerometer located on the afterbody records the vibration response shown. [Reprinted with permission from R. L. Piziali and C. D. Mote, Jr., "The Snow Ski as a Dynamic System," J. Dynamic Syst. Meas. Control, Trans. ASME **94**, 134 (1972).]

Page 63: Natural frequency with "good and bad vibrations"

[David A. Lind et Scott P. Sanders, The Physics of Skiing: Skiing at the Triple Point, 2nd edition; 2013]

One way to kill bad vibrations?

Control your skis!

As Jean-Pierre?

Use passively controls
[L. Rothemann, H. Schretter, Active vibration damping of the alpine ski; 2010]

How to do it actively?
Need to control a PDE, with finite energy,
that is with saturating controls.

As Jean-Pierre would do!

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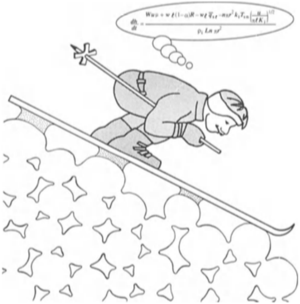

FIGURE 2.1. This skier heads down the hill, his skis lubricated by a film of water that forms under his skis. In his thoughts he mulls over a mathematical formula that we will discuss later in Chapter 8 on snow friction processes. (Colbeck, 1992. Drawn by Marilyn Aber, CRREL.)

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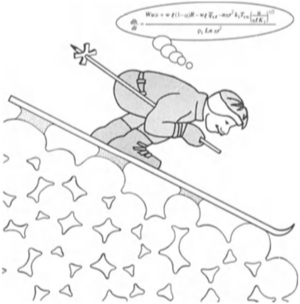

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Given a PDE, there exists now a large variety on methods to design linear controllers. It is well known that saturation can reduce the performance or even destabilize the system, even for finite-dimensional systems.
More precisely, even if

$$\dot{z} = Az + BKz \tag{1}$$

is asymp. stable, it may hold that

$$\dot{z} = Az + \mathtt{sat}(BKz) \tag{2}$$

is not globally asymptotically stable.
It may exist new equilibrium, new limit cycles...
See e.g. [Tarbouriech, Garcia, Gomes da Silva Jr., Queinnec; 2011]

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Stability issues : a finite-dimensional example

Saturating a stabilizing feedback law can lead to **instabilities**.

An illustrative example

$$\frac{dz}{dt} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} z + \begin{bmatrix} 0 \\ -1 \end{bmatrix} u$$

Open-loop eigenvalues: $\lambda_1 = 1$, $\lambda_2 = -1$. Setting $u = Kz$ with $K = \begin{bmatrix} 13 & 7 \end{bmatrix}$, the origin is globally asymptotically stable.

Considering $u = \mathtt{sat}(Kz)$ with saturation level $u_s = 5$, we get

1. $z_0 = [-2 \;\; -3]^\top$: the trajectory converges to $z^\star = [-5 \; 0]^\top$;
2. $z_0 = [-3 \;\; -3]^\top$: the trajectory diverges.

Stability issues : a finite-dimensional example

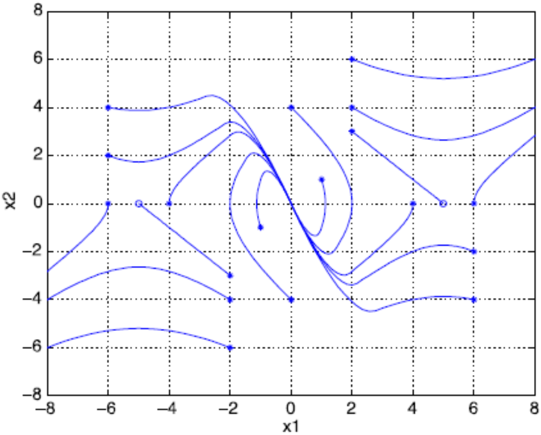

Figure: (*): initial conditions, (o): equilibrium points

Goal of this talk:

What happens if in (2), instead of matrices A, B..., we have operators? More precisely, what happens if A generates a semigroup and B is a bounded control operator? An example of such a nonlinear PDE given by (2):

Wave equation with saturating in-domain control

Two objectives

- Well-posedness
- Stability

of the wave equation in presence of a disturbed saturating control with a Lyapunov method.

[Haraux; 18], [Martinez; 99], [Martinez and Vancostenoble; 00], [Alabau-Boussouira; 12]

Outline

1. Well-posedness and stability of linear wave equation with a saturated in-domain control

   Lyapunov method, LaSalle invariance principle

2. Design of a strict Lyapunov function for L^2 saturation

   Robustness result

3. With localized (L^∞) saturation

   strict Lyapunov method, robustness result

4. With non-monotone damping

   comparison with a linear time-varying equation

5. Conclusion

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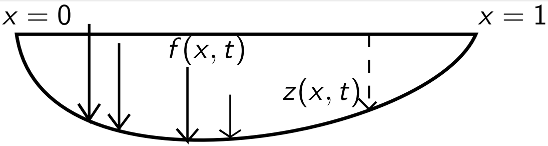

1D wave equation with in-domain control.

Dynamics of the vibration:

$$z_{tt}(x,t) = z_{xx}(x,t) + f(x,t),\ \forall x \in (0,1), t \geq 0, \qquad (3)$$

Boundary conditions, $\forall t \geq 0$,

$$\begin{aligned} z(0,t) &= 0\ , \\ z(1,t) &= 0\ , \end{aligned} \qquad (4)$$

and with the following initial condition, $\forall x \in (0,1)$,

$$\begin{aligned} z(x,0) &= z^0(x)\ , \\ z_t(x,0) &= z^1(x)\ , \end{aligned} \qquad (5)$$

where z^0 and z^1 stand respectively for the initial deflection and the initial deflection speed.

When closing the loop with a linear control

Let us define the linear control by

$$f(x,t) = -az_t(x,t), x \in (0,1), \ \forall t \geq 0, \qquad (6)$$

and consider the energy

$$E = \frac{1}{2}\int (z_x^2 + z_t^2)dx.$$

Formal computation. Along the solutions to (3), (4) and (6):

$$\begin{aligned} \dot{E} &= \int_0^1 (z_x z_{xt} - az_t^2 + z_t z_{xx})dx \\ &= -\int_0^1 az_t^2 dx + [z_t z_x]_{x=0}^{x=1} \\ &= -\int_0^1 az_t^2 dx \end{aligned}$$

Thus, it $a > 0$, E is a (non strict) Lyapunov function.

Using standard technics (Lumer-Philipps thereom (for the well-posedness) and Huang-Prüss theorem (for the exp. stability)):

Proposition

$\forall a > 0$, $\forall (z^0, z^1)$ in $H := H_0^1(0,1) \times L^2(0,1)$,
$\exists$! solution (z, z_t): $[0, \infty) \to H$ to (3)-(6). Moreover, $\exists$ C, $\mu > 0$, such that, for any initial condition H, it holds, $\forall t \geq 0$,

$$\|z\|_{H_0^1(0,1)} + \|z_t\|_{L^2(0,1)} \leq Ce^{-\mu t}(\|z^0\|_{H_0^1(0,1)} + \|z^1\|_{L^2(0,1)}).$$

In the previous proposition:

- stability
- attractivity of the equilibrium
- with an exponential speed

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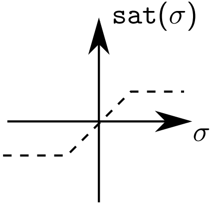
Let us consider now the nonlinear control

$$f(x,t) = -\mathtt{sat}(az_t(x,t)),\ x \in (0,1),\ \forall t \geq 0, \tag{7}$$

where sat is the localized saturated map:



$$\mathtt{sat}(\sigma) = \begin{cases} \sigma & \text{if } |\sigma| < 1 \\ \mathtt{sign}(\sigma) & \text{else} \end{cases}$$

Equation (3) in closed loop with the control (7) becomes

$$z_{tt} = z_{xx} - \mathtt{sat}(az_t) \tag{8}$$

A formal computation gives, along the solutions to (8) and (4),

$$\dot{E} = -\int_0^1 z_t \mathtt{sat}(az_t)dx$$

which asks to handle the nonlinearity $z_t\mathtt{sat}(az_t)$.

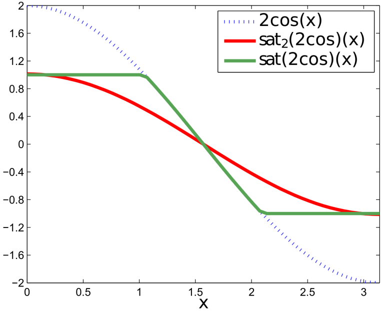
Remark: Choice of the saturation map

[Slemrod; 89] and [Lasiecka, Seidman; 03] deal with L^2 saturation:
Given $\sigma : [0,1] \to \mathbb{R}$, $\mathtt{sat}_2(\sigma)$ is the function defined by

$$\mathtt{sat}_2(\sigma)(x) = \begin{cases} \sigma(x) & \text{if } \|\sigma\|_{L^2(0,1)} < 1 \\ \frac{\sigma(x)}{\|\sigma\|_{L^2(0,1)}} & \text{else} \end{cases}$$

Here we consider **localized** saturation which is more physically relevant:

$$\mathtt{sat}(\sigma(x)) = \begin{cases} \sigma(x) & \text{if } |\sigma(x)| < 1 \\ \mathtt{sign}(\sigma(x)) & \text{else} \end{cases}$$



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with $\sigma = 2\cos$

Well-posedness of this nonlinear PDE

Theorem 1 [CP, Tarbouriech, Gomes da Silva Jr; 2016]

$\forall a \geq 0$, for all (z^0, z^1) in $(H^2(0,1) \cap H^1_0(0,1)) \times H^1_0(0,1)$, there exists a unique (strong) solution z: $[0,\infty) \to H^2(0,1) \cap H^1_0(0,1)$ to (8) with the boundary conditions (4) and the initial condition (5).

Consider

$$A_1 \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ u_{xx} - \mathtt{sat}(av) \end{pmatrix}$$

with the domain $D(A_1) = (H^2(0,1) \cap H^1_0(0,1)) \times H^1_0(0,1)$.
Let us use a generalization of Lumer-Phillips theorem which is the so-called Crandall-Liggett theorem, as given in [Barbu; 1976]. See also [Brezis; 1973] and [Miyadera; 1992].
Again two conditions

1. A_1 is dissipative, that is

$$\Re\left(\left\langle A_1 \begin{pmatrix} u \\ v \end{pmatrix} - A_1 \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}\right\rangle\right)_H \leq 0$$

2. For all $\lambda > 0$, $D(A_1) \subset \mathtt{Ran}(I - \lambda A_1)$

First item: Easy step!

Instead of proving $\Re\left(\langle A_1\begin{pmatrix} u \\ v \end{pmatrix} - A_1\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}\rangle\right)_H \leq 0$, let us check, for all $\begin{pmatrix} u \\ v \end{pmatrix} \in H \quad (= H_0^1(0,1) \times L^2(0,1))$:

$$\Re\left(\langle A_1\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix}\rangle\right)_H \leq 0$$

To do that, using the definition of A_1, and of the scalar product in $H_0^1(0,1) \times L^2(0,1)$, it is equal to:

$$\begin{aligned} &\int_0^1 v_x(x)\overline{u_x(x)}dx + \int_0^1 (u_{xx}(x) - \mathtt{sat}(a\,v(x)))\overline{v(x)}dx \;, \\ =& \int_0^1 v_x(x)\overline{u_x(x)}dx + \int_0^1 u_{xx}(x)\overline{v(x)}dx - \int_0^1 \mathtt{sat}(a\,v(x))\overline{v(x)}dx \\ =& [u_x(x)\overline{v(x)}]_{x=0}^{x=1} - \int_0^1 \mathtt{sat}(a\,v(x))\overline{v(x)}dx \leq 0 \end{aligned}$$

due to the boundary and since $a \geq 0$.

Second item asks to deal with a nonlinear ODE.

Let $\begin{pmatrix} u \\ v \end{pmatrix} \in H$ we have to find $\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} \in D(A_1)$ such that

$$(I - \lambda A_1) \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$$

that is

$$\begin{cases} \tilde{u} - \lambda \tilde{v} = u \ , \\ \tilde{v} - \lambda(\tilde{u}_{xx} - \mathtt{sat}(a\,\tilde{v})) = v \ , \end{cases}$$

In particular, we have to find $\tilde{u}$ such that

$$\begin{aligned} &\tilde{u}_{xx} - \frac{1}{\lambda^2}\tilde{u} - \mathtt{sat}(\frac{a}{\lambda}(\tilde{u} - u)) = -\frac{1}{\lambda}v - \frac{1}{\lambda^2}u \\ &\tilde{u}(0) = \tilde{u}(1) = 0 \end{aligned}$$

holds.

Nonhomogeneous nonlinear ODE with two boundary conditions

Lemma

If a is nonnegative and λ is positive, then there exists $\tilde{u}$ solution to

$$\begin{array}{c} \tilde{u}_{xx} - \frac{1}{\lambda^2}\tilde{u} - \mathtt{sat}(\frac{a}{\lambda}(\tilde{u} - u)) = -\frac{1}{\lambda}v - \frac{1}{\lambda^2}u \\ \tilde{u}(0) = \tilde{u}(1) = 0 \end{array} \tag{9}$$

To prove this lemma, let us introduce the following map

$$\begin{array}{rccc} \mathcal{T}: & L^2(0,1) & \to & L^2(0,1) \ , \\ & y & \mapsto & z = \mathcal{T}(y) \ , \end{array}$$

where $z = \mathcal{T}(y)$ is the unique solution to

$$\begin{array}{c} z_{xx} - \frac{1}{\lambda^2}z = -\frac{1}{\lambda}v - \frac{1}{\lambda^2}u + \mathtt{sat}(\frac{a}{\lambda}(y - u)) \ , \\ z(0) = z(1) = 0 \ . \end{array}$$

Prove that $\mathcal{T}$ is well defined and apply the Schauder fixed-point theorem (see e.g., [Coron; 2007]), to deduce that there exists y such that $\mathcal{T}(y) = y$

$\tilde{u} = y$ solves (9)

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Theorem 2

$\forall a > 0$, for all (z^0, z^1) in $(H^2(0,1) \cap H_0^1(0,1)) \times H_0^1(0,1)$, the solution to (8) with the boundary conditions (4) and the initial condition (5) satisfies the following stability property, $\forall t \geq 0$,

$$\|z(.,t)\|_{H_0^1(0,1)} + \|z_t(.,t)\|_{L^2(0,1)} \leq \|z^0\|_{H_0^1(0,1)} + \|z^1\|_{L^2(0,1)} \ ,$$

together with the attractivity property

$$\|z(.,t)\|_{H_0^1(0,1)} + \|z_t(.,t)\|_{L^2(0,1)} \to 0, \quad \text{as } t \to \infty \ .$$

Due to Theorem 1, the formal computation

$$\dot{E} = -\int_0^1 z_t \mathtt{sat}(az_t)dx$$

makes sense. This is only a weak Lyapunov function $\dot{E} \leq 0$
(the state is (z, z_t), and there is no $-z^2$).

To be able to apply LaSalle's Invariance Principle, we have to check that the trajectories are precompact
(see e.g. [Dafermos, Slemrod; 1973]).

It comes from:

Lemma

The canonical embedding from $D(A_1)$, equipped with the graph norm, into $H_0^1(0,1) \times L^2(0,1)$ is compact.

Sketch of the proof of

The canonical embedding from $D(A_1)$, equipped with the graph norm, into $H_0^1(0,1) \times L^2(0,1)$ is compact.

Consider a sequence $\begin{pmatrix} u_n \\ v_n \end{pmatrix}_{n\in\mathbb{N}}$ in $D(A_1)$, which is bounded with the graph norm, that is $\exists M > 0$, $\forall n \in \mathbb{N}$,

$$\begin{aligned} \left\| \begin{pmatrix} u_n \\ v_n \end{pmatrix} \right\|^2_{D(A_1)} &:= \left\| \begin{pmatrix} u_n \\ v_n \end{pmatrix} \right\|^2 + \left\| A_1 \begin{pmatrix} u_n \\ v_n \end{pmatrix} \right\|^2, \\ &= \int_0^1 (|u_n'|^2 + |v_n|^2 + |v_n'|^2 \\ &\quad + |u_n'' - a\,\mathrm{sat}(v_n)|^2) dx < M \end{aligned}$$

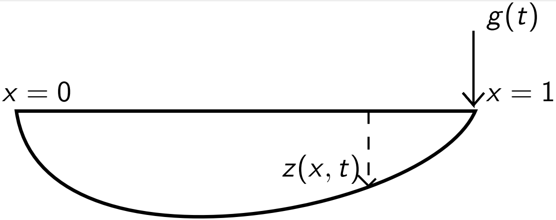
From that, we deduce that $\int_0^1 (|v_n|^2 + |v_n'|^2) dx$ and $\int_0^1 (|u_n'|^2 + |u_n''|^2) dx$ are bounded.
Thus there exists a subsequence which converges in $H_0^1(0,1) \times L^2(0,1)$.

Using the dissipativity of A_1, and previous lemma the trajectory $\begin{pmatrix} z(.,t) \\ z_t(.,t) \end{pmatrix}$ is precompact in $H_0^1(0,1) \times L^2(0,1)$.

Moreover the ω-limit set $\omega \left[\begin{pmatrix} z(.,0) \\ z_t(.,0) \end{pmatrix} \right] \subset D(A_1)$, is not empty and invariant with respect to the nonlinear semigroup $T(t)$ (see [Slemrod; 1989]).

We now use LaSalle's invariance principle to show that $\omega \left[\begin{pmatrix} z(.,0) \\ z_t(.,0) \end{pmatrix} \right] = \{0\}$.

Therefore the convergence property holds. □



1D wave equation with a boundary control.

Dynamics: $\forall x \in (0,1), t \geq 0,$

$$z_{tt}(x,t) = z_{xx}(x,t),$$

Boundary conditions: $\forall t \geq 0,$

$$\begin{aligned} z(0,t) &= 0 \;, \\ z_x(1,t) &= -\mathtt{sat}(bz_t(1,t)) \;, \end{aligned}$$

In the same work, stability proof using the sector condition + strict Lyapunov function.

▸ Wave equation and saturated boundary control

2 – Strict Lyapunov function

For the wave equation+ saturated in-domain control, a non-strict Lyapunov function has been computed.

- Thus a priori no robustness margin.
  What happens in presence of noise?
- For linear PDE, we have exponential convergence (see Proposition on Slide 10).
  Do we have exp. stability for the nonlinear PDE?

Rewrite the wave equation as a abstract control system:

$$\begin{cases} \frac{dz}{dt} = Az + Bu, \\ z(0) = z_0. \end{cases}$$

There exists a self-adjoint and pos. def. $P \in \mathcal{L}(H)$ s.t.

$$\langle (A - BB^\star)z, Pz, \rangle_H + \langle Pz, (A - BB^\star)z \rangle_H \leq -\|z\|_H^2, \quad \forall z \in D(A) \tag{10}$$

In presence of saturated input and disturbances

Consider the L^2 saturated case

$$\begin{cases} \dfrac{dz}{dt} = Az - B\mathtt{sat}_2(B^\star z), \\ z(0) = z_0, \end{cases} \tag{11}$$

In presence of saturated input and disturbances

Consider the L^2 saturated case + disturbance

$$\begin{cases} \dfrac{dz}{dt} = Az - B\mathtt{sat}_2(B^\star z + \underline{d}), \\ z(0) = z_0, \end{cases} \tag{11}$$

where $d : (0, \infty) \to L^2(0,1)$ is a disturbance.

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Recall the L^2 saturation: Given $u : [0,1] \to \mathbb{R}$, $\mathtt{sat}_2(\sigma)$ is the function defined by $\mathtt{sat}_2(\sigma) = \begin{cases} \sigma & \text{if } \|\sigma\|_{L^2(0,1)} < 1 \\ \frac{\sigma}{\|\sigma\|_{L^2(0,1)}} & \text{else} \end{cases}$

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What can be said about the exp. stability when $d = 0$ and about the robustness in presence of d?

ISS notion

Input-to-State Stability (ISS) definition

A positive definite function $V : H \to \mathbb{R}_{\geq 0}$ is said to be an ISS-Lyapunov function with respect to d if $\exists$ two class $\mathcal{K}_\infty$ functions α and ρ such that, for any solution to (11)

$$\frac{d}{dt} V(z) \leq -\alpha(\|z\|_H) + \rho(\|d\|_{L^2(0,1)}).$$

Remark: Of course ISS Lyapunov function
$+ \exists$ two functions $\underline{\alpha}$ and $\overline{\alpha}$ of class[1] $\mathcal{K}$ such that

$$\underline{\alpha}(\|z\|_H) \leq V(z) \leq \overline{\alpha}(\|z\|_H) \;, \forall z \in H$$

$\Rightarrow$ the origin of (11) with $d = 0$ is globally asymptotically stable.

[1] $\alpha : [0, \infty) \to [0, \infty)$ is of class $\mathcal{K}$ if it is continuous, zero at zero and increasing. It is of class $\mathcal{K}_\infty$ if it is moreover unbounded.

Input-to-state stability result

Theorem 3 [Marx, Chitour, CP; to appear]

Suppose that Assumption 1 holds and let $P \in \mathcal{L}(H)$ be a self-adjoint and positive operator satisfying (10). Then, there exists M such that

$$V(z) := \langle Pz, z \rangle_H + M\|z\|_H^3 \qquad (12)$$

is an ISS-Lyapunov function for (11).

The proof follows the finite-dimensional case considered in [Liu, Chitour, and Sontag; 1996].

Sketch of the proof

Let us consider the following candidate Lyapunov function

$$V(z) := \langle Pz, z\rangle_H + M\|z\|_H^3$$

Along the strong solutions to (11), with $\tilde{A} = A - BB^\star$

$$\frac{d}{dt}\langle Pz, z\rangle_H = \langle Pz, Az\rangle_H + \langle PAz, z\rangle_H \\ + 2\langle PB(\mathtt{sat}_2(B^\star z) - \mathtt{sat}_2(B^\star z + d)), z\rangle_H$$

Sketch of the proof

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$$V(z) := \langle Pz, z\rangle_H + M\|z\|_H^3$$

Along the strong solutions to (11), with $\tilde{A} = A - BB^\star$

$$\frac{d}{dt}\langle Pz, z\rangle_H = \langle Pz, \tilde{A}z\rangle_H + \langle P\tilde{A}z, z\rangle_H$$
$$+ 2\langle PB(B^\star z - \mathtt{sat}_2(B^\star z), z\rangle_H$$
$$+ 2\langle PB(\mathtt{sat}_2(B^\star z) - \mathtt{sat}_2(B^\star z + d)), z\rangle_H$$

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Along the strong solutions to (11), with $\tilde{A} = A - BB^\star$

$$\begin{aligned}\frac{d}{dt}\langle Pz, z\rangle_H = {} & \langle Pz, \tilde{A}z\rangle_H + \langle P\tilde{A}z, z\rangle_H \\ & + 2\langle PB(B^\star z - \mathtt{sat}_U(B^\star z), z\rangle_H \\ & + 2\langle PB(\mathtt{sat}_2(B^\star z) - \mathtt{sat}_2(B^\star z + d)), z\rangle_H \\ \leq & -\|z\|_H^2 + 2\|B^\star z\|_{L^2(0,1)}\|P\|_{\mathcal{L}(H)}\|B^\star z - \mathtt{sat}_2(B^\star z)\|_{L^2(0,1)} \\ & + 2\langle \mathtt{sat}_2(B^\star z) - \mathtt{sat}_2(B^\star z + d), B^\star Pz\rangle_{L^2(0,1)},\end{aligned}$$

Sketch of the proof

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Along the strong solutions to (11), with $\tilde{A} = A - BB^\star$

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using $\mathtt{sat}_2$ Lipchitz, Cauchy-Schwarz inequality and the fact that $B^\star$ is bounded.

Moreover using $\|d\|_{L^2(0,1)}\|z\|_H \leq \varepsilon\|d\|^2_{L^2(0,1)} + \frac{1}{\varepsilon}\|z\|^2_H$ and $\|B^\star z - \mathtt{sat}_2(B^\star z)\|_{L^2(0,1)} \leq \langle \mathtt{sat}_2(B^\star z), B^\star z\rangle_{L^2(0,1)}$, we get

$$\begin{aligned}\frac{d}{dt}\langle Pz, z\rangle_H \leq & -\left(1 - \frac{\|B^\star\|^2_{\mathcal{L}(H,L^2(0,1))}\|P\|^2_{\mathcal{L}(H)}}{\varepsilon_1}\right)\|z\|^2_H \\ & +2\|B^\star\|_{\mathcal{L}(H,L^2(0,1))}\|P\|_{\mathcal{L}(H)}\|z\|_H\langle \mathtt{sat}_2(B^\star z), B^\star z\rangle_{L^2(0,1)} \\ & +k^2\varepsilon_1\|d\|^2_{L^2(0,1)}\end{aligned}$$

where ε_1 is a positive value that will be selected later.

Thus

$$\frac{d}{dt}W(z) \leq \text{good term} + \text{bad term} + d^2$$

Secondly, using the dissipativity of the operator $A_{\mathtt{sat}}$, $\langle \mathtt{sat}_2(B^\star z) - \mathtt{sat}_2(B^\star z + d), B^\star z\rangle_{L^2(0,1)} \leq C_0 \|d\|_{L^2(0,1)}$, and $\|z\|_H \|d\|_{L^2(0,1)} \leq \frac{1}{\varepsilon_2}\|z\|_H^2 + \varepsilon_2 \|d\|^2_{L^2(0,1)}$, one has

$$
\begin{aligned}
\frac{2M}{3}\frac{d}{dt}\|z\|_H^3 =& M\|z\|\big(\langle Az, z\rangle_H + \langle z, Az\rangle_H\big) \\
& - 2M\|z\|_H \langle B\mathtt{sat}_2(B^\star z + d), z\rangle_H \\
\leq & - 2M\|z\|_H\big(\langle \mathtt{sat}_2(B^\star z), B^\star z\rangle_{L^2(0,1)} \\
& + \langle \mathtt{sat}_2(B^\star z) - \mathtt{sat}_2(B^\star z + d), B^\star z\rangle_{L^2(0,1)}\big) \\
\leq & - 2M\|z\|_H \langle \mathtt{sat}_2(B^\star z), B^\star z\rangle_{L^2(0,1)} \\
& + 2MC_0\|z\|_H\|d\|_{L^2(0,1)} \\
\leq & -2M\|z\|_H \langle \mathtt{sat}_2(B^\star z), B^\star z\rangle_{L^2(0,1)} \\
& + \frac{2MC_0}{\varepsilon_2}\|z\|_H^2 + 2MC_0\varepsilon_2\|d\|^2_{L^2(0,1)},
\end{aligned}
$$

where ε_2 is a positive value that has to be selected. For an appropriate choice of M, ε_1 and ε_2 we deduce the result. □

Secondly, using the dissipativity of the operator $A_{\mathtt{sat}}$, $\langle \mathtt{sat}_2(B^\star z) - \mathtt{sat}_2(B^\star z + d), B^\star z\rangle_{L^2(0,1)} \leq C_0\|d\|_{L^2(0,1)}$, and $\|z\|_H\|d\|_{L^2(0,1)} \leq \frac{1}{\varepsilon_2}\|z\|_H^2 + \varepsilon_2\|d\|_{L^2(0,1)}^2$, one has

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\frac{2M}{3}\frac{d}{dt}\|z\|_H^3 =& M\|z\|\big(\langle Az, z\rangle_H + \langle z, Az\rangle_H\big) \\
& - 2M\|z\|_H\langle B\mathtt{sat}_2(B^\star z + d), z\rangle_H \\
\leq & - 2M\|z\|_H\big(\langle \mathtt{sat}_2(B^\star z), B^\star z\rangle_{L^2(0,1)} \\
& + \langle \mathtt{sat}_2(B^\star z) - \mathtt{sat}_2(B^\star z + d), B^\star z\rangle_{L^2(0,1)}\big) \\
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\end{aligned}
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where ε_2 is a positive value that has to be selected. For an appropriate choice of M, ε_1 and ε_2 we deduce the result. □

What happens with *sat* instead of sat_2? What is the speed of convergence of

$$\begin{cases} \frac{d}{dt} z = Az - B\mathtt{sat}(B^\star z), \\ z(0) = z_0, \end{cases} \tag{13}$$

Theorem

Hence, the origin of (13) is semi-globally exponentially stable in $D(A)$, that is for any positive r and any z_0 in $D(A)$ satisfying $\|z_0\|_{D(A)} \leq r$, there exist two positive constants $\mu := \mu(r)$ and $K := K(r)$ such that

$$\|W_\sigma(t)z_0\|_H \leq Ke^{-\mu t}\|z_0\|_H, \quad \forall t \geq 0. \tag{14}$$

Remarks • on Korteweg-de Vries equation: [Rosier, Zhang; 2006] and [Marx, Cerpa, CP, Andrieu; 2017].

We may deduce a global asymptotic stability (but without any estimation of the convergence speed).

• In our work the monotonicity is crucial and also only 1D

See [Martinez, Vancostenoble; 2000] for $N \leq 2$. See also last part of this presentation for $N = 1$.

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Sketch of the proof

Let $\tilde{V}(z)$ be the Lyapunov function candidate defined by

$$z \in D(A) \mapsto \tilde{V}(z) := \langle Pz, z\rangle_H + \tilde{M}\|z\|_H^2,$$

where $\tilde{M} > 0$ will be selected later. As before, using the dissipativity of the operator $A - B^\star B$, one has

$$\frac{d}{dt}\tilde{M}\|z\|_H^2 \leq -2\tilde{M}\langle B^\star z, \mathtt{sat}(B^\star z)\rangle_{L^2(0,1)}. \tag{15}$$

and

$$\frac{d}{dt}\langle Pz, z\rangle_H \leq -\|z\|_H^2 + 2\langle B^\star Pz, B^\star z - \mathtt{sat}(B^\star z)\rangle_{L^2(0,1)}.$$

The term $2\langle B^\star Pz, B^\star z - \mathtt{sat}(B^\star z)\rangle_{L^2(0,1)}$ is "controlled" differently.

Consider $r > 0$ and a strong solution for (13), whose initial condition $z_0 \in D(A)$ is such that

$$\|z_0\|_{D(A)} \leq r$$

First note that, from the dissipativity, it implies $\|z(t)\|_{D(A)} \leq r$ for all $t \geq 0$.

$$\begin{aligned} &|\langle B^\star Pz, B^\star z - \mathtt{sat}(B^\star z)\rangle_{L^2(0,1)}| \\ &\qquad \leq \|B^\star Pz\|_{L^\infty(0,1)}\|B^\star z - \mathtt{sat}(B^\star z)\|_{L^1(0,1)} \\ &\qquad \leq C\|Pz\|_{D(A)}\|B^\star z - \mathtt{sat}(B^\star z)\|_{L^1(0,1)} \\ &\qquad \leq C'\|P\|_{\mathcal{L}(D(A))}\|z\|_{D(A)}\langle \mathtt{sat}(B^\star z), B^\star z\rangle_{L^2(0,1)} \end{aligned}$$

Therefore

$$\begin{aligned} \frac{d\tilde{V}}{dt} &\leq -\|z\|_H^2 - 2(\tilde{M} - C'\|P\|_{\mathcal{L}(D(A))}\|z\|_{D(A)})\langle \mathtt{sat}(B^\star z), B^\star z\rangle \\ &\leq -\|z\|_H^2 - 2(\tilde{M} - C'\|P\|_{\mathcal{L}(D(A))}r)\langle \mathtt{sat}(B^\star z), B^\star z\rangle \\ &\leq -\|z\|_H^2 \end{aligned}$$

for a suitable $\tilde{M}$. The result follows.

Consider again the controlled wave equation:

$$\begin{cases} z_{tt} = z_{xx} + u, \quad (t,x) \in \mathbb{R}_+ \times [0,1] \\ z(t,0) = z(t,1) = 0, \quad t \in \mathbb{R}_+ \\ z(0,x) = z_0(x),\ z_t(0,x) = z_1(x), \quad x \in [0,1], \end{cases}$$

Nonlinear damping σ law given by the damping

$u(t,x) = -\sqrt{a(x)}\sigma(\sqrt{a(x)}z_t(t,x))$ where $\forall x \in \omega, a_0 < a(x) \leq a_\infty, a_0 >$

References

[Martinez; 99], [Martinez, Vancostenoble; 00]

Question

What about nonmonotone nonlinearities σ?

Nonmonotone damping

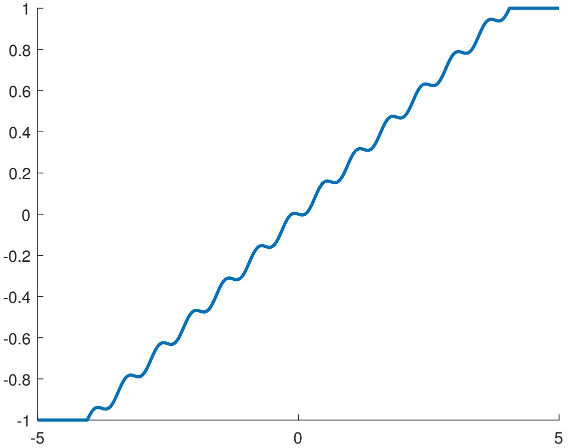
Nonmonotone damping

A function σ is a nonmonotone damping if

1. it is locally Lipschitz
2. $\sigma(0) = 0$
3. for all $s \in \mathbb{R}$, $\sigma(s)s > 0$
4. the function σ is differentiable at $s = 0$ with $\sigma'(0) = C_1$, where C_1 is a positive constant.

Nonmonotone damping

for example: $\sigma(s) = \mathtt{sat}\left(\frac{1}{4}s - \frac{1}{30}\sin(10s)\right)$



Regularity issues

Since the function σ is (possibly) nonmonotone, then the LaSalle's Invariance Principle does not apply !

Moreover, the classical functional setting

$$H = H_0^1(0,1) \times L^2(0,1),$$

is not sufficient to ensure a L^∞ regularity for the state z_t.

Solution (inspired by [Haraux; 2009])

Our solution consists in using the functional setting

$$H_p := \left(W^{1,p}(0,1) \cap H_0^1(0,1)\right) \times L^p(0,1),$$

where $p \in [1, \infty]$.

$$\begin{cases} z_{tt} = z_{xx} - \sqrt{a(x)}\sigma(\sqrt{a(x)}z_t),\ (t,x) \in \mathbb{R}_+ \times [0,1] \\ z(t,0) = z(t,1) = 0,\ t \in \mathbb{R}_+ \\ z(0,x) = z_0(x),\ z_t(0,x) = z_1(x),\ x \in [0,1]. \end{cases} \tag{Sys}$$

Theorem [Chitour, Marx, CP; under submission] (well-posedness)

$\forall$ initial condition $(z_0, z_1) \in H_\infty$, $\exists!$ solution
$(z, z_t) \in L^\infty(\mathbb{R}_+; W^{1,\infty}(0,1)) \times W^{1,\infty}(\mathbb{R}_+; L^\infty(0,1))$ to (Sys).
Moreover, one has

$$\|(z, z_t)\|_{H_\infty(0,1)} \leq 2\max\left(\|z_0'\|_{L^\infty(0,1)}, \|z_1\|_{L^\infty(0,1)}\right)$$

Main results

$$\begin{cases} z_{tt} = z_{xx} - \sqrt{a(x)}\sigma(\sqrt{a(x)}z_t),\ (t,x) \in \mathbb{R}_+ \times [0,1] \\ z(t,0) = z(t,1) = 0,\ t \in \mathbb{R}_+ \\ z(0,x) = z_0(x),\ z_t(0,x) = z_1(x),\ x \in [0,1]. \end{cases} \tag{Sys}$$

Theorem [Chitour, Marx, CP; under submission] (convergence)

Given $r > 0$. Consider initial conditions in H_∞ satisfying

$$\|(z_0, z_1)\|_{H_\infty} \leq r\ .$$

Then, $\forall p \in [2, \infty)$, $\exists\ K := K(r)$ and $\mu := \mu(r)$ such that

$$\|(z, z_t)\|_{H_p} \leq K e^{-\mu t} \|(z_0, z_1)\|_{H_p},\ \forall t \geq 0.$$

Well-posedness proof (1)

- Fixed-point theorem $\Rightarrow$ existence and uniqueness in $[0, T]$.
- The estimate is proved thanks to the following result

Theorem [Haraux; 2009]

Let us consider initial condition in H_∞. Let us introduce the following functional

$$\phi(z, z_t) = \int_0^1 [F(z - z_t) + F(z + z_t)]dx,$$

where F is any even and convex function. Then, the time derivative of ϕ along the trajectories of (Sys) satisfies

$$\frac{d}{dt}\phi(z, z_t) \leq 0.$$

Well-posedness proof (2)

Due to the latter theorem, one has $\phi(z, z_t) \leq \phi(z_0, z_1)$, for all $t \geq 0$. Then, the result follows by setting

$$F(s) := [\mathrm{Pos}(|s| - 2\max(\|z_0'\|_{L^\infty(0,1)}, \|z_1\|_{L^\infty(0,1)})],$$

where

$$\mathrm{Pos}(s) := \begin{cases} s \text{ if } s > 0, \\ 0 \text{ if } s \leq 0. \end{cases}$$

This implies that $\phi(z, z_t) = 0$ and then, for all $t \geq 0$

$$\|(z, z_t)\|_{H_\infty(0,1)} \leq 2\max\left(\|z_0'\|_{L^\infty(0,1)}, \|z_1\|_{L^\infty(0,1)}\right).$$

Question

What about the asymptotic stability ?

Steps

Consider (Sys), with initial conditions in H_∞. Thanks to this regularity:

1. Prove the result in $H_2 = H_0^1(0,1) \times L^2(0,1)$
2. Deduce the result in H_p by an interpolation theorem (Riesz-Thaurin theorem), with

$$H_p = \left(W^{1,p}(0,1) \cap H_0^1(0,1)\right) \times L^p(0,1)$$

Strategy

Transforming the nonlinear time-invariant system as a trajectory of a linear time-variant system.

A detour via linear time-variant systems

System (Sys) can be seen as a trajectory of a linear time-variant system (LTV).

$$\begin{cases} z_{tt} = z_{xx} - a(x)d(t,x)z_t, & (t,x) \in \mathbb{R}_+ \times [0,1], \\ z(t,0) = z(t,1) = 0, & t \in \mathbb{R}_+, \\ z(0,x) = z_0(x),\ z_t(0,x) = z_1(x), & x \in [0,1], \end{cases} \qquad \text{(LTV-wave)}$$

where

$$d(t,x) = \begin{cases} \dfrac{\sigma(\sqrt{a(x)}z_t)}{\sqrt{a(x)}z_t}, & \sqrt{a(x)}z_t \neq 0, \\ C_1, & \sqrt{a(x)}z_t = 0, \end{cases}$$

where $C_1 = \sigma'(0)$.

Abstract system

Let us recall that $H_2 = H^1_0(0,1) \times L^2(0,1)$ and let us introduce $U = L^2(0,1)$. Consider the abstract system

$$\begin{cases} \frac{d}{dt} y = Ay - d(t)BB^\star y := A_d(t)y, \\ y(\tau) = y_\tau, \tau \geq 0, \end{cases} \qquad \text{(Abstract)}$$

with $y = \begin{bmatrix} z & z_t \end{bmatrix}^\top$, $A : D(A) \subset H_2 \to H_2$ defined as

$$A = \begin{bmatrix} 0 & I_{H_2} \\ \partial_{xx} & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 & \sqrt{a(x)} I_{H_2} \end{bmatrix}^\top,$$

with $D(A) = (H^2(0,1) \cap H^1_0(0,1)) \times H^1_0(0,1)$.

Trajectories

(Sys) and (Abstract) share one trajectory, i.e. when $\tau = 0$.

Proposition (for convergence result)

Suppose that there exist $d_0, d_1 > 0$ such that

$$d_0 \leq d(t) \leq d_1.$$

Then, if

$$\begin{cases} \frac{d}{dt} y = Ay - d_0 BB^\star y := A_{d_0} y, \\ y(0) = y_0, \end{cases}$$

is exponentially stable, the trajectory of (Abstract) with $\tau = 0$ converges to 0.

Lyapunov proof of this proposition

Exponential stability $\Rightarrow \exists\ \widehat{P} \in \mathcal{L}(H_2)$ and $C > 0$ such that

$$\langle \widehat{P} y, A_{d_0} \rangle_{H_2} + \langle \widehat{P} A_{d_0} y, y \rangle_{H_2} \leq -C \|y\|^2_{H_2}$$

Time derivative of the Lyapunov functional

$$\widehat{V}(y) := \langle \widehat{P} y, y \rangle_{H_2} + \widehat{M} \|y\|^2_{H_2}$$

along the trajectories of (Abstract) with $\widehat{M} = \frac{2(d_1 - d_0)\|\widehat{P}\|_{\mathcal{L}(H_2)}}{d_0 \|B\|_{\mathcal{L}(H_2, U)}}$,

$$\frac{d\widehat{V}}{dt}(y) \leq -C\|y\|^2_{H_2}$$

Then,

$$\|y\|^2_{H_2} \leq \frac{\|\widehat{P}\|_{\mathcal{L}(H_2)} + \widehat{M}}{\widehat{M}} \exp\left(-\frac{C}{\|\widehat{P}\|_{\mathcal{L}(H_2)} + \widehat{M}} t \right) \|y_0\|^2_{H_2},\ \forall t \geq 0$$

Back to the proof of convergence result

Recall that

$$d(t,x) = \begin{cases} \dfrac{\sigma(\sqrt{a}(x) z_t)}{\sqrt{a(x)} z_t}, & \sqrt{a(x)} z_t \neq 0, \\ C_1, & \sqrt{a(x)} z_t = 0, \end{cases}$$

and that

$$\|(z, z_t)\|_{H_\infty(0,1)} \leq 2 \max \left(\|z_0'\|_{L^\infty(0,1)}, \|z_1\|_{L^\infty(0,1)} \right) \leq 2r,$$

then

$$d_0 := \min_{\xi \in [-2\sqrt{a_\infty} r, 2\sqrt{a_\infty} r]} \frac{\sigma(\xi)}{\xi} \leq d(t,x)$$

$$\leq \max_{\xi \in [-2\sqrt{a_\infty} r, 2\sqrt{a_\infty} r]} \frac{\sigma(\xi)}{\xi} := d_1.$$

Then, one can prove easily that

$$\|(z, z_t)\|_{H_2} \leq K(r) e^{-\mu(r) t} \|(z_0, z_1)\|_{H_2}$$

which is the result.

5 – Conclusion and further research lines

Results

1. Asymptotic stability in H_p for non-monotone damping
2. Semi-global exponential stability in H for monotone damping
3. Instead of wave equations, abstract operator theories could be developped

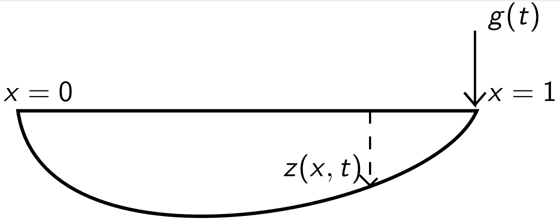
Further research lines

1. What about quasilinear hyperbolic systems

$$\begin{cases} z_t + \Lambda(z) z_x = 0 \\ z(t,0) = H z(t,1) + B u(t)? \end{cases}$$

   See [Coron, Ervedoza, Ghoshal, Glass, Perrollaz; 17], and the current work of M. Dus for BV solutions.

2. N-dimensional wave equations ?
   $N \leq 2$ in [Martinez, Vancostenoble; 2000]



1D wave equation with a boundary control.

Dynamics:

$$z_{tt}(x,t) = z_{xx}(x,t),\ \forall x \in (0,1), t \geq 0, \qquad (16)$$

Boundary conditions, $\forall t \geq 0$,

$$\begin{aligned} z(0,t) &= 0 , \\ z_x(1,t) &= g(t) , \end{aligned} \qquad (17)$$

and with the same initial condition, $\forall x \in (0,1)$,

$$\begin{aligned} z(x,0) &= z^0(x) , \\ z_t(x,0) &= z^1(x) . \end{aligned} \qquad (18)$$

When closing the loop with a linear boundary control

Let us define the linear control by

$$g(t) = -bz_t(1,t),\ x \in (0,1),\ \forall t \geq 0 \qquad (19)$$

and consider

$$E_2 = \frac{1}{2}\int (e^{\mu x}(z_t + z_x)^2 dx + \int (e^{-\mu x}(z_t - z_x)^2 dx,$$

Formal computation. Along the solutions to (16), (17) and (19):

$$\dot{E}_2 \;=\; -\mu E_2 + \tfrac{1}{2}\left(e^{\mu}(1-b)^2 - e^{-\mu}(1+b)^2\right) z_t^2(1,t)$$

Assuming $b > 0$ and letting $\mu > 0$ such that $e^{\mu}(1-b)^2 \leq e^{-\mu}(1+b)^2$, it holds $\dot{E}_2 \leq -\mu E_2$ and thus E_2 is a strict Lyapunov function and thus (16)-(19) is exponentially stable.

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When closing the loop with a saturating control

Let us consider now the nonlinear control $g(t) = -\mathtt{sat}(bz_t(1,t)), \forall t \geq 0$. The boundary conditions become:

$$z(0,t) = 0 \ , \quad z_x(1,t) = -\mathtt{sat}(bz_t(1,t)) \ . \tag{20}$$

Theorem (stability with boundary control)

$\forall b > 0$, for all (z^0, z^1) in $\{(u,v),\ (u,v) \in H^2(0,1) \times H^1_{(0)}(0,1),\ u_x(1) + \mathtt{sat}(bv(1)) = 0,\ u(0) = 0\}$, the solution to (16) with the boundary conditions (20) and the initial condition (5) satisfies the following stability property, $\forall t \geq 0$,

$$\|z(.,t)\|_{H^1_{(0)}(0,1)} + \|z_t(.,t)\|_{L^2(0,1)} \leq \|z^0\|_{H^1_{(0)}(0,1)} + \|z^1\|_{L^2(0,1)} \ ,$$

together with the attractivity property

$$\|z(.,t)\|_{H^1_{(0)}(0,1)} + \|z_t(.,t)\|_{L^2(0,1)} \to 0, \ \text{ as } t \to \infty \ .$$

To prove the well-posedness of the Cauchy problem we prove that A_2 defined by

$$A_2 \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ u'' \end{pmatrix}$$

with the domain $D(A_2) = \{(u, v),\ (u, v) \in H^2(0,1) \times H^1_{(0)}(0,1),\ u'(1) + \mathtt{sat}(bv(1)) = 0,\ u(0) = 0\}$ is a semigroup of contraction.

The global stability property comes directly from the dissipativity of A_2.

The global attractivity property comes from the following lemma:

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The global stability property comes directly from the dissipativity of A_2.

The global attractivity property comes from the following lemma:

Lemma (semi-global exponential stability)

For all $r > 0$, there exists $\mu > 0$ such that, for all initial condition satisfying

$$\|z^{0\prime\prime}\|^2_{L^2(0,1)} + \|z^1\|^2_{H^1_{(0)}(0,1)} \leq r^2 \ , \tag{21}$$

it holds

$$\dot{E}_2 \leq -\mu E_2$$

along the solutions to (16) with the boundary conditions (20).

Sketch of the proof of this lemma

First note that by dissipativity of A_2, it holds that

$$t \mapsto \left\| A_2 \begin{pmatrix} z(.,t) \\ z_t(.,t) \end{pmatrix} \right\|_H$$

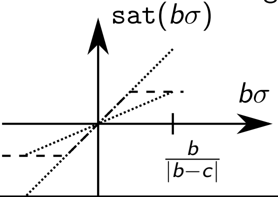
is a non-increasing function. Thus, for all $t \geq 0$,

$$|z_t(1,t)| \leq \left\| A_2 \begin{pmatrix} z(.,0) \\ z_t(.,0) \end{pmatrix} \right\|_H \quad .$$

Now for all initial conditions satisfying (21), there exists $c \neq b$ such that, for all $t \geq 0$,

$$(b-c)|z_t(1,t)| \leq 1$$

and thus the following local sector condition holds:



Letting $\sigma = z_t(1,t)$, it holds

$$(\mathtt{sat}(b\sigma) - b\sigma)(\mathtt{sat}(b\sigma) - (b-c)\sigma) \leq 0$$

We come back to the Lyapunov function candidate E_2. Given $b > 0$, using the previous inequality, we compute

$$\begin{aligned}
\dot{E}_2 &= -\mu E_2 + e^{\mu}(\sigma - \mathtt{sat}(b\sigma))^2 - e^{-\mu}(\sigma + \mathtt{sat}(b\sigma))^2 \\
&\leq -\mu E_2 + \begin{pmatrix} \sigma \\ \mathtt{sat}(b\sigma) \end{pmatrix}^{\top} \begin{pmatrix} e^{\mu} - e^{-\mu} - b^2(b-c) & -e^{\mu} - e^{-\mu} + b + b(b-c) \\ -e^{\mu} - e^{-\mu} + b + b(b-c) & -1 + e^{\mu} - e^{-\mu} \end{pmatrix} \\
&\qquad \times \begin{pmatrix} \sigma \\ \mathtt{sat}(b\sigma) \end{pmatrix} \\
&\leq -\mu E_2
\end{aligned}$$

with a suitable choice of constant values μ and c.
The semi-global exponential stability follows. □

▸ Back to the wave equation with in-domain control

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$$\begin{array}{rcl} \dot{E}_2 & = & -\mu E_2 + e^{\mu}(\sigma - \mathtt{sat}(b\sigma))^2 - e^{-\mu}(\sigma + \mathtt{sat}(b\sigma))^2 \\ & \leq & -\mu E_2 + \begin{pmatrix} \sigma \\ \mathtt{sat}(b\sigma) \end{pmatrix}^\top \begin{pmatrix} e^{\mu} - e^{-\mu} - b^2(b-c) & -e^{\mu} - e^{-\mu} + b + b(b-c) \\ -e^{\mu} - e^{-\mu} + b + b(b-c) & -1 + e^{\mu} - e^{-\mu} \end{pmatrix} \\ & & \quad \times \begin{pmatrix} \sigma \\ \mathtt{sat}(b\sigma) \end{pmatrix} \\ & \leq & -\mu E_2 \end{array}$$

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The semi-global exponential stability follows. □

Back to the wave equation with in-domain control