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by
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Ô moun païs,

Ô TOULOUSE !

Museum St Raymond

St Raymond

Jean-Pierre Raymond

Phd with Marc Atteia
in Toulouse, of course...

60 + publications

25 ? thesis

Involvement in many exterior activities (in addition to his local activities),
in particular, relations with Tunisia and India (Tata Institute Bangalore)

Many collaborations

Non Convex Calculus of Variations

$$(\mathcal{P}) \inf\{\int_\Omega (g(\nabla u) + f(x,u))dx,\ u \in W_0^{1,p}(\Omega) + \Phi\}$$

To have weak l.s.c., we need a hypothesis of quasiconvexity on g.
g is quasiconvex if

$$\forall A \in M^{N,N},\ \forall \varphi \in C_0^\infty(\Omega),\ \int_\Omega g(A + \nabla\varphi)dx \geq \int_\Omega g(A)dx.$$

Relaxed problem

$$(\mathcal{P}R) \inf\{\int_\Omega (g^{**}(\nabla u) + f(x,u))dx,\ u \in W_0^{1,p}(\Omega) + \Phi\}$$

Regularized problem
θ_ϵ regularizing sequence, $\quad g_\epsilon^{**} = g^{**} \star \theta_\epsilon$

$$(\mathcal{P}R_\epsilon) \inf\{\int_\Omega (g_\epsilon^{**}(\nabla u) + f(x,u) + \epsilon|\nabla u|^2)dx,\ u \in W_0^{1,p}(\Omega) + \Phi\}$$

Sufficient condition of optimality :
If u is solution of $(\mathcal{P}R)$ and if
$meas\{x \in \Omega,\ g^{**}(\nabla u(x)) \neq g(\nabla u(x))\} = 0$, then u is solution of $(\mathcal{P})$.
Here :
No hypothesis of convexity for g.
Natural hypothesis on g and f, in particular coercivity of g (at infinity) in $W_0^{1,p}$.
Affinity hypothesis (natural) :
$K = \{V \in \mathbf{R}^N,\ g^{**}(V) < g(V)\} = \cup_{i=1}^m K_i$, g^{**} is affine on K_i.
Ω is uniformly convex and Φ is Lipschitz

Theorem

If $\forall u \in \mathbf{R}$, $f_u(x,u) \neq 0$, then $(\mathcal{P})$ admits Lipschitz solutions.

Existence result without convexity hypothesis.
Analogous result for vector valued functions u.

Method

-Obtention of bounded Lipschitz solutions for $(\mathcal{P}R_\epsilon)$.
-Obtention of Lipschitz solutions for $(\mathcal{P}R)$.
-Regularity in H^1_{loc} of adjoint solutions for Lipschitz solutions of $(\mathcal{P}R)$.
If u is solution of $(\mathcal{P}R)$, from Euler equation, there exists $q \in L^{p'}(\Omega)^N$ such that

$$\nabla g^{**}(\nabla u(x)) = q(x)\, a.e.,\ \operatorname{div} q = f_u(., u) \,\text{in}\, \mathcal{D}'(\Omega).$$

Adjoint solution

$$-\int_\Omega \operatorname{div} \xi(x) u(x) dx = \int_\Omega g^*_{q_j}(q(x))\xi_j(x) dx$$

$$\forall \xi \in L^{p'}(\Omega)^N,\ \text{dist}(\partial\Omega, \text{Supp}\xi_i) > 0,\ \operatorname{div} \xi \in L^{p'}(\Omega),\ g^*_{q_j}(q) \in \partial g^*(q).$$

$$\operatorname{div} q = f_u(., u) \,\text{in}\, \mathcal{D}'(\Omega).$$

If u is a Lipschitz solution of $(\mathcal{P}R)$,

$$E_i = \{x \in \Omega,\ u \text{ is differentiable at } x,\ \nabla u(x) \in K_i\}.$$

Then

$$\exists \gamma_i \in \mathbf{R}^N \text{ such that } \forall x \in E_i,\ \nabla g^{**}(\nabla u(x)) = \gamma_i = q(x).$$

As $q \in H^1_{loc}$, we have $\operatorname{div} q(x) = 0$, *a.e.* in E_i.
But $\operatorname{div} q(x) = f_u(x, u)x)) \neq 0$. This leads to a contradiction except if $meas(E_i) = 0$.
Then

$$meas\{x,\ g^{**}(\nabla u(x)) \neq g(\nabla u(x))\} = 0$$

and u is solution of $(\mathcal{P})$.

Optimal control problem with state constraints

Optimal control problem for semilinear parabolic equations with state constraints.

Case of a priori unbounded controls.

Work in collaboration with H. Zidani.

$$\begin{aligned}
&\frac{\partial y}{\partial t} - \Delta y + f(x,t,y) = 0, \text{ in } \Omega \times (0,T) = Q,\\
&\frac{\partial y}{\partial \nu} + g(s,t,y,v) = 0, \text{ on } \Gamma \times (0,T) = \Sigma,\\
&y(0) = w.
\end{aligned}$$

v, w : controls, $v \in V_{ad} \subset L^{\sigma}(\Sigma)$, $w \in W_{ad} \subset C(\bar{\Omega})$.
$\Phi(y) \in C \subset C(\bar{D})$ closed and convex.

$$(\mathcal{P}) \inf\{J(y,v,w),\, (y,v,w) \in C(\bar{Q}) \times V_{ad} \times W_{ad},\, \Phi(y) \in C\}$$

$$J(y,v,w) = \int_Q F(x,t,y)dxdt + \int_\Sigma G(s,t,y,v)dsdt + \int_\Omega L(x,y(T),w)dx.$$

A priori unbounded controls and

$$V_{ad} = \{v \in L^{\sigma}(\Sigma),\ v(s,t) \in K_v(s,t)\, a.e.\}$$

with K_v multimapping with non empty closed convex values in the subsets of **R**.
Regularity for f, g, Φ, F, G, L, f and g not too much non monotone (derivatives bounded from below).
Hypothesis of strong stability :

$$C_\gamma = \{\varphi \in C(\bar{D}),\ \inf_{z \in C} ||\varphi - z||_{C(\bar{D})} \leq \gamma.$$

$(\mathcal{P}_\gamma)$: inf on C_γ.
Hypothesis :
$\exists \bar{\epsilon}$, $\exists \bar{r}$, $\forall \gamma' \in [\gamma, \gamma + \bar{\epsilon}]$, $\inf(\mathcal{P}_\gamma) - \inf(\mathcal{P}_{\gamma'}) \leq \bar{r}(\gamma' - \gamma)$.
Existence of optimal control $(\bar{v}, \bar{w})$ is assumed corresponding to $\bar{y}$.
Look for optimality conditions in the form of Pontryagin maximum principle via an extension of Ekeland variational principle.

Theorem

$\exists \bar{p} \in L^1(0,T;W^{1,1}(\Omega))$, $\exists \bar{\nu} \in \mathbf{R}$, $\exists \bar{\mu} \in \mathcal{M}(\bar{D}), \exists \tilde{\Sigma} \subset \Sigma$, *such that (without strong stability)*

$$(\bar{\nu},\bar{\mu}) \neq 0,\ \bar{\nu} \geq 0,\ (\bar{\mu}, z - \Phi(\bar{y}) \leq 0 \, \forall z \in C,$$
$$-\frac{\partial \bar{p}}{\partial t} - \Delta \bar{p} + f'_y \bar{p} = \bar{\nu} F'_y + [\Phi'(y)^* \bar{\mu}]_Q,$$
$$\frac{\partial \bar{p}}{\partial \nu} + g'_y \bar{p} - \bar{\nu} G'_y + [\Phi'(y)^* \bar{\mu}]_\Sigma,$$
$$\bar{p}(T) = \bar{\nu} L'_y + [\Phi'(y)^* \bar{\mu}]_{\Omega_T},$$
$$H_\Sigma(s,t,\bar{y},\bar{v},\bar{p},\bar{\nu}) = \min v \in K_v(s,t) H_\Sigma(s,t,\bar{y}.v,\bar{p},\bar{\nu}) \forall (s,t) \in \tilde{\Sigma},$$
$$meas(\tilde{\Sigma}) = meas(\Sigma),$$
$$\int_\Omega \bar{\nu} L'_w(\bar{w} - w) + < \bar{p}(0) + [\Phi'(y)^* \bar{\mu}]_{\Omega_0}, \bar{w} - w > \leq 0,\ forall w \in W_{ad},$$

with

$$H_\Sigma(s,t,y,v,p,\nu) = \nu G(s,t,y,v) - pg(s,t,y,v).$$

With strong stability, we can take $\bar{\nu} = 1$.

Method ; To find a good setting to be able to apply the Ekeland variational principle to a penalized functional. Impossible to use it directly with unbounded controls because the Ekeland metric does not imply convergence in a Lebesgue space.
Ekeland variational principle : (X, d) a metric space and $J : X \to [0, +\infty]$ l.s.c and not identically $+\infty$.
$x \in X$ such that $J(x) \leq \inf J + \epsilon$. Then $\forall \delta$, $\exists y \in X$ such that $J(y) \leq J(x)$, $d(x, y) \leq \delta$ and $\forall z \in X \setminus \{y\}$, $J(y) \leq J(z) + \frac{\epsilon}{\delta} d(z, y)$.
Here the authors take

$$V_{ad}(\tilde{v}, k) = \{v \in V_{ad},\, |v(s,t) - \tilde{v}(s,t)| \leq k \text{ a.e. on } \Sigma\}$$

$$d((v_1, w_1), (v_2, w_2)) = \text{meas}(\{(s,t)\, v_1(s,t) \neq v_2(s,t)\}) + ||w_1 - w_2||_\infty .$$

Then $(V_{ad}(\tilde{v}, k) \times W_{ad}, d)$ is a complete metric space and J is continuous.

They construct admissible perturbations of approximate optimal solutions via a "diffuse perturbation" the existence of which is proved using the convexity Lyapunov theorem.

They obtain Pontryagin maximum principle by exploiting optimality conditions for the approximate problems

Details of proofs is quite delicate and use very fine notions of integration and measure theory.

Null controllability of a fluid-structure model

Work in collaboration with M. Vanninathan.
They consider Stokes system on a 2-d fixed domain Ω of annular type with boundary $\Gamma_e \cup \Gamma_i$, $\Gamma_e \cap \Gamma_i = \emptyset$, coupled with a structure which can only have translations but can vibrate like a N-finite dimensional approximation of an elasticity system.

$$y' - \Delta y + \nabla \pi = u.\xi_{omega} \text{ in } \Omega \times (0, T),$$
$$\operatorname{div} y = 0$$
$$y = 0 \text{ in } \Gamma e \times (0, T),$$
$$y(0) = y^0,$$
$$y = Mq' \text{ i n } \Gamma_i \times (0, T),$$
$$q'' + Aq = -\int_{\Gamma_i} M^T \sigma(y, n) n d\sigma,$$
$$q(0) = q^0, \; q'(0) = q^1.$$

For simplicity , $M = I$, $N = 2$ and $A = I$.

Theorem

$\forall y^0 \in L^2(\Omega)$, $\operatorname{div} y^0 = 0$, $\forall q^0 \in \mathbf{R}^2$, $q^1 \in \mathbf{R}^2$ with $y^0.n = q^1.n$ on Γ_i and $Y^0.n = 0$ on Γ_e, there exists $u \in L^2(Q)$ such that

$$y(T) = 0,\ q(T) = 0,\ q'(T) = 0.$$

Method : proofs are based on an observability inequality for the adjoint system which is obtained from ad'hoc global Carleman estimates.
Adjoint system (after time reversal)

$$\begin{aligned}
&\Phi' - \Delta\Phi + \nabla p = 0 \text{ in } \Omega \times (0, T),\\
&\operatorname{div}\Phi = 0,\\
&\Phi = r' \text{ i n} \Gamma_i \times (0, T),\ \Phi = 0 \text{ in } \Gamma e \times (0, T),\\
&r'' + r = -\int_{\Gamma_i} \sigma(\Phi, p) n\\
&\Phi(0 = \Phi^0,\ r(0) = r^0,\ r'(0) = r^1.
\end{aligned}$$

Several steps :

(i) Carleman estimate for the fluid (Φ) in function of ∇p with local term in Φ on ω.

(ii) Treatment of boundary terms.

(iii) Carleman estimate on the pressure (p) with local term on the pressure and trace of the pressure.

(iv) Estimate on the trace of the pressure involving a term on the structure.

(v) Estimate on the structure (r) using the coupling relation on Γ_i.

(vi) Null controllability with a ficiticious control on the divergence because of the local term on the pressure with a regularity result on this control.

(vii) Elimination of the ficticious control thanks to the regularity of this control.

Feedback boundary stabilization of the 3-d Navier Stokes equation

Unstable solution of stationary Navier-Stokes equation (w, ξ).

$$\begin{aligned} &-\nu \Delta w + (w.\nabla)w + \nabla \xi = f \text{ in } \Omega, \\ &\operatorname{div} w = 0, \\ &w = u_s^\infty \text{ on } \Gamma. \end{aligned}$$

Now $(z, q) = (w + y, \xi + p)$ is solution of the Navier-Stokes equations with initial value $w + y_0$ (and suitable boundary conditions including a control) if

$$\begin{aligned} &\frac{\partial y}{\partial t} - \nu \Delta y + (y.\nabla)w + (w.\nabla)y + (y.\nabla)y + \nabla p = 0 \text{ in } \Omega, \\ &\operatorname{div} y = 0, \\ &y = Mu \text{ on } \Gamma, \\ &y(0) = y_0. \end{aligned}$$

One wants to find the control u in the form of a feedback located on a part of Γ so that the system is stable (exponentially stable ...) for y_0 small enough.

Strategy : obtain a feedback for the linearized problem and use it for the nonlinear one.

Linearized problem :

$$\frac{\partial y}{\partial t} - \nu \Delta y + (y.\nabla)w + (w.\nabla)y + \nabla p = 0 \text{ in } \Omega,$$
$$\operatorname{div} y = 0,$$
$$y = Mu \text{ on } \Gamma,$$
$$y(0) = y_0.$$

Rewrite the system using the Leray projector P

$$Ay = \nu P\Delta y - P(y.\nabla)w - P(w.\nabla)y,\ A_0 y = \nu P\Delta y,$$

$$BMu = (\lambda_0 I - A)PD_A Mu,$$

D_A : Dirichlet operator associated with A, λ_0 in the resolvent set of A

$$Mu = m.u - \frac{m}{\int_\Gamma m}(\int_\Gamma m(u.n))n$$

with

$$m = 1 \text{ on } \Gamma_0,\ m = 0 \text{ on } \Gamma \setminus \Gamma_c, \Gamma_0 \subset \Gamma_c \subset \Gamma.$$

$$Py' = APy + BMu.$$

Write $Py = y$ for simplicity.

Method :

Auxiliary optimal control problem

$$\inf\{I(,y,u),\ u\in L^2(V^0)\},\ y'=Ay+BMu,\ \ y(0)=y_0\},$$

where

$$I(y,u)=\frac{1}{2}\int_0^{+\infty}\int_\Omega |(-A_0)^{-\frac{1}{2}}y|^2 dxdt+\frac{1}{2}\int_0^{+\infty}|u(t)|^2_{V^0(\Gamma)}dt.$$

There exists an operator Π (in suitable spaces) such that the optimal control satisfies

$$\Pi^*=\Pi,\ \ u=MB^*\Pi y$$

and Π is solution of the algebraic Riccati equation

$$A^*\Pi+\Pi A-\Pi BM^2B^*\Pi+(-A_0)^{-1}=0$$

and this feedback stabilizes exponentially the problem.
This is done for initial data satisfying compatibility conditions.

For weaker initial data which do not satisfy compatibility conditions, JPR uses a cut-off $\theta(t)$ on an interval $(0, t_0)$ and the feedback operator depends on time on this interval and then satifies the algebraic Riccati equation for $t \geq t_0$.
After that, the nonlinear problem is treated like the linear problem with a source term and the feedback for the linear problem (with source term) is applied to obtain the exponential stability.
This work requires a very fine and precise analysis on the regularity of the solutions. It gives a completely correct answer to this question of stabilization for the Navier-Stokes system. This question has been treated by other authors, but....not in a completely correct way....

Jean-Pierre,

Welcome to the club.

You will enjoy something fantastic....

Freedom !!