# Conference on
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Thermal fluid Stabilization

Mythily Ramaswamy
Chennai Mathematical Institute,
Chennai, India

# Collaborators and Financial Support

Collaborators:

Jean-Pierre Raymond (IMT, Toulouse),
Arnab Roy (Post-Doc, INRIA Nancy, France)

Praveen C (TIFR CAM) ,
Ruchi Sandilya. (Weistrass Institute, Berlin).

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# contents

# Boussinesq System

Model : Heat conducting incompressible fluid in

$\Omega$, a two dimensional polygonal domain.

- Fluid enters through one side and flows out through opposite side.
- Rest of the boundary is a wall, thermally insulated.

Governing equations, Boussinesq system

with mixed boundary conditions of Dirichlet and Neumann type

describes

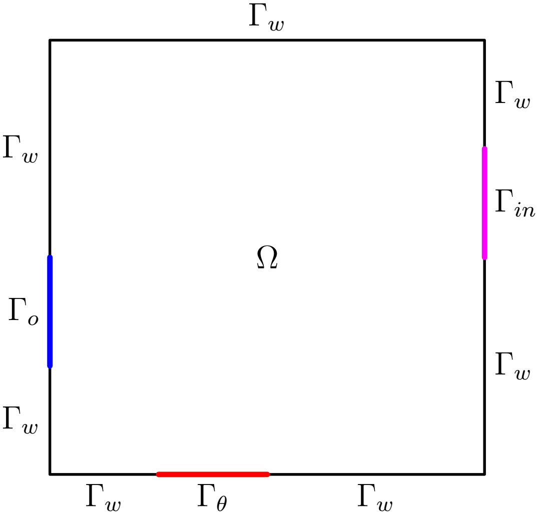
evolution of velocity, pressure, temperature of the fluid,

# Boussinesq System

Aim :

- To stabilize the flow around an unstable steady state of the system, controlling velocity and temperature of the fluid at the inflow boundary, $\Gamma_c$.
- To get the boundary control in feedback form in a suitable finite dimensional space.
- To implement it numerically .

# Physical set up



# Mathematical set up

Stationary state temperature, velocity and pressure $(\tau_s, \mathbf{w}_s, q_s)$ satisfies

$$\begin{cases} -\kappa\Delta\tau_s + \mathbf{w}_s.\nabla\tau_s \;=\; g_s \text{ in } \Omega, \\ \tau_s \;= 0 \;\text{ on }\; \Gamma_c, \quad \dfrac{\partial\tau_s}{\partial n} \;=\; 0 \;\text{ on }\; \Gamma\setminus\Gamma_c, \\ (\mathbf{w}_s.\nabla)\mathbf{w}_s - \operatorname{div}(\nu(\nabla\mathbf{w}_s + (\nabla\mathbf{w}_s)^T) - q_s I) \;= \beta\;\tau_{\mathbf{s}}, \\ \operatorname{div}\mathbf{w}_s \;=\; 0 \;\text{ in }\; \Omega; \quad \mathbf{w}_s \;=\; 0 \text{ on } \Gamma_w, \\ \mathbf{w}_s \;= \mathbf{z}_s \text{ on } \Gamma_c\;, \quad (\nu(\nabla\mathbf{w}_s + (\nabla\mathbf{w}_s)^T) - q_s I)\; n \;=\; 0 \;\text{ on }\; \Gamma_n, \end{cases} \tag{1}$$

Denote Cauchy stress tensor as

$$\sigma(\mathbf{w}, q) = \nu(\nabla\mathbf{w} + (\nabla\mathbf{w})^T) - qI.$$

# Controlled evolution equation

The flow near the steady state describes the evolution of $(\tau, \mathbf{u}, p)$, the deviation from the steady state. This controlled system, with boundary controls $\tau_c, \mathbf{u}_c$ is

$$
\begin{cases}
\dfrac{\partial \tau}{\partial t} - \kappa \Delta \tau \ + \ \mathbf{w}_s \cdot \nabla \tau \ + \ \mathbf{u} \cdot \nabla \tau_s \ + \ \mathbf{u} \cdot \nabla \tau \ = \ 0 \ \text{ in } \ Q, \quad \tau(0) \ = \ \tau_0 \\
\tau \ = \ \tau_c \ \text{ on } \Gamma_c, \ \dfrac{\partial \tau}{\partial n} \ = \ 0 \ \text{ on } \ \Gamma \setminus \Gamma_c, \\
\dfrac{\partial \mathbf{u}}{\partial t} - \operatorname{div} \sigma(\mathbf{u}, p) + (\mathbf{w}_s . \nabla)\mathbf{u} + (\mathbf{u}.\nabla)\mathbf{w}_s + (\mathbf{u}.\nabla)\mathbf{u} \ = \beta \ \tau, \\
\operatorname{div} \mathbf{u} = 0, \ \text{ in } \ Q, \quad \mathbf{u} = \mathbf{u_c} \text{ on } \Gamma_c, \\
\mathbf{u} = 0 \text{ on } \Gamma_w, \ \sigma(\mathbf{u}, p)\mathbf{n} = 0 \text{ on } \Gamma_n, \quad \mathbf{u}(0) = \mathbf{u}_0 \text{ in } \Omega,
\end{cases}
\tag{2}
$$

# Earlier Results

- Nguyen and Raymond (2015) : Boundary Stabilisation of Navier-Stokes Equations.
  Mixed boundary conditions, with Dirichlet-Neumann junction angle less than or equal to $\frac{\pi}{2}$.
  Solution in $\mathbf{H}^{3/2+\delta}$.

- John Burns, X. He , Weiwei Hu. (2016) : Feedback stabilization of a thermal fluid system with mixed boundary control.
  Model for Energy efficient buildings
  Dirichlet-Neumann junction angle less than or equal to $\pi$ but they used the results from the above paper.
  Treatment of Dirichlet boundary conditions by approximations through Robin type boundary conditions.

# Main issues

Aim :

For a given $\omega > 0$, find finite dimensional boundary controls for temperature and velocity $(\tau_c, \mathbf{u_c})$ in feedback form, such that the solution decays at the rate of $e^{-\omega t}$.

Main Issues:

- Loss of regularity for solutions of elliptic equations in polygonal domain with mixed boundary conditions
- A suitable function space to define the Boussinesq operator and to tackle the nonlinearity
- Stabilizability of the Boussinesq system under mixed boundary conditions
- Construction of finite dimensional feedback controls on the boundary

# Loss of regularity for the solution

For Boussinesq system with mixed boundary conditions, if compatibility conditions hold at junction points :

$$(\mathbf{u}, \tau) \in \mathbf{H}^2_\delta(\Omega) \times H^2_\delta(\Omega), \ \forall \quad 1/2 < \delta < 1.$$

Use the imbedding $H^s_\delta \subset H^{s-\delta}, \ \forall \quad 0 \leq \delta \leq s.$

Conclude that the solution

$$(\mathbf{u}, \tau) \in \mathbf{H}^{3/2-\delta} \times H^{3/2-\delta}, \ \forall \quad 0 < \delta < 1/2.$$

Boussinesq system with nonhomogeneous boundary conditions : If

$$(\mathbf{u_c}, \tau_c) \in \mathbf{H}^{1/2}_{00}(\Gamma_c) \times H^{1/2}_{00}(\Gamma_c),$$

then solution $(\mathbf{u}, \tau) \in \mathbf{H}^{3/2-\delta} \times H^{3/2-\delta}$ for $0 < \delta < 1/2$.

# Consequences and Solution Procedures

- Normal trace in negative Sobolev spaces only.
- Define suitably using variational formulation the operator $(A, D(A))$.

$$D(\mathcal{A}) = \{(\theta, \mathbf{v}) \in H^1_{\Gamma_c}(\Omega) \times \mathbf{V}^1_{\Gamma_d}(\Omega) : (\xi, \phi) \mapsto a((\theta, \mathbf{v}), (\xi, \phi)) \text{ is } Z \text{ continuous}\},$$

$Z = L^2(\Omega) \times \mathbf{V}^0_{n,\Gamma_d}(\Omega)$ and
$\forall\, (\theta, \mathbf{v}) \in D(\mathcal{A})$, $(\xi, \phi) \in H^1_{\Gamma_c}(\Omega) \times \mathbf{V}^1_{\Gamma_d}(\Omega)$,

$$\langle (\lambda_0 I - \mathcal{A})(\theta, \mathbf{v}), (\xi, \phi) \rangle_Z = a((\theta, \mathbf{v}), (\xi, \phi)).$$

- Deduce Operator formulation of the control problem

$$\mathbf{Y}' = (\mathcal{A} + \omega I)\mathbf{Y} + Bv.$$

# Consequences and Solution Procedures

- Use appropriate weighted function spaces to get Green's formula adapted to polygonal domains
- Identify of the adjoint operators
- Check Hautus condition for stabilizability for Boussinesq system with mixed boundary conditions
- Identify a suitable finite dimensional function space on the boundary for defining boundary control
- Usual estimates to treat the nonlinear term not applicable
- Adapt them suitably after Identifying the function space to set up a fixed point iteration
- Get the solution of the nonlinear closed loop system with feedback boundary control.

# Operator formulation of control problem

The problem : To stabilize the system

$$\mathbf{Y}' = (\mathcal{A} + \omega I)\mathbf{Y} + Bv.$$

in $L^2(\Omega) \times \mathbf{V}^0_{n,\Gamma_d}(\Omega)$ with decay rate $\omega > 0$ using a finite dimensional feedback control.

Find $N_\omega \in \mathbb{N}^*$ such that

$$.... < Re\lambda_{N_\omega+1} < -\omega < Re\lambda_{N_\omega} \leq Re\lambda_{N_\omega-1} \leq .... \leq Re\lambda_1,$$

where $\lambda_j$ are the eigenvalues of $\mathcal{A}$, repeated according to their multiplicity.

# Construction of the Control

Seek a finite dimensional control in the form

$$\tau_c(x,t) = \sum_{i=1}^{N} f_i(t) g_\theta^i(x), \quad \mathbf{u_c}(x,t) = \sum_{i=1}^{N} f_i(t) \mathbf{g_v^i}(x), \qquad (3)$$

for $g_\theta^i$ and $\mathbf{g_v^i}$ localized in $\Gamma_c$ and $f_i$ are scalar functions.

Choice of the family $\mathbf{g}_i(x)$

- Introduce the space $(E^*(\lambda_j))_{1\leq j\leq N_\omega}$ = unstable eigenspaces of $\mathcal{A}$
- Define a subspace of $L^2(0,T;L^2(\Gamma_c)\times \mathbf{L}^2(\Gamma_c))$,

$$U_0 = \cup_{j=1}^{N_\omega}(\mathrm{Re}\, B^* E^*(\lambda_j) \cup \mathrm{Im}\, B^* E^*(\lambda_j)),$$

- Take for some $N$, $\{\tilde{\mathbf{g}}_1, \tilde{\mathbf{g}}_2, ..., \tilde{\mathbf{g}}_N\}$ as a basis of $U_0$
- Take $\mathbf{g}_i = m(g_\tau^i, \mathbf{g_v^i})$, for $m$, smooth function with compact support in $\Gamma_c$.

# Construction of the Control

- Define the operator $\mathcal{B} \in \mathcal{L}(\mathbb{R}^N, (D(\mathcal{A}^*))')$:

$$\mathcal{B}v = \sum_{i=1}^{N} v_i B\mathbf{g}_i.$$

- The system now is

$$\mathbf{Y}' = \mathcal{A}\mathbf{Y} + \mathcal{B}v.$$

- For this choice of actuators, show that the system is stabilizable by checking Hautus condition.

# Construction of the Control

- Project the system onto unstable subspace and stable subspace.
- For the finite dimensional system projected onto unstable subspace, find the feedback control by solving a finite dimensional Riccati equation
- The operator $\mathcal{K} = -\mathcal{B}^*\mathcal{P}$ from $Z$ to $\mathbb{R}^n$, provides a stabilizing feedback for $(\mathcal{A} + \omega I, \mathcal{B})$.
- This feedback control stabilizes the full linear system

# Regularity of the solutions

Regularity result for the closed loop linearized system with source terms:

If the initial condition $(\theta_0, \mathbf{v}_0) \in H^\varepsilon(\Omega) \times \mathbf{V}^\varepsilon_{n,\Gamma_d}(\Omega)$ and

the source term $(f_1, \mathbf{f}_2) \in L^2(0,\infty; H^{-1+\varepsilon}_{\Gamma_c}(\Omega)) \times L^2(0,\infty; \mathbf{H}^{-1+\varepsilon}_{\Gamma_d}(\Omega))$,

then the solution $(\theta, \mathbf{v})$ has two parts:

- one belonging to $L^2(0,\infty;\ D((\lambda_0 I - \mathcal{A})^{\frac{1}{2}+\frac{\varepsilon}{2}})) \cap H^{\frac{1}{2}+\frac{\varepsilon}{2}}(0,\infty;\ Z)$ corresponding to the initial condition and
- the other part to $H^1(0,\infty; H^{\frac{3}{2}-\varepsilon}(\Omega) \times \mathbf{H}^{\frac{3}{2}-\varepsilon}(\Omega))$, corresponding to the finite dimensional feedback control.

# Stabilization of the Nonlinear system

**Theorem**

*Let* $\varepsilon \in (0, 1/2)$. *For a given* $\omega > 0$, *there exist*

- *a finite family of actuators,* $\mathbf{g}_i = (g_{\theta,i}, \mathbf{g}_{\mathbf{v},i}) \in H^{3/2}_{00}(\Gamma_c) \times \mathbf{H}^{3/2}_{00}(\Gamma_c)$
- *and positive constants* $\mu_0, C_0$

*such that if* $(\theta_0, \mathbf{v}_0) \in H^{\varepsilon}(\Omega) \times \mathbf{V}^{\varepsilon}_{n,\Gamma_d}(\Omega)$, $\mu \in (0, \mu_0)$ *with*

$$||(\theta_0, \mathbf{v}_0)||_{H^{\varepsilon}(\Omega) \times \mathbf{V}^{\varepsilon}_{n,\Gamma_d}(\Omega)} \leq C_0 \mu,$$

*then the closed loop system admits a unique solution in the ball* $B_\mu = \left\{(\theta, \, \mathbf{v}) \in X \ : \ ||e^{\omega t}(\theta, \mathbf{v})||_X \leq \mu\right\}$. *Also the solution satisfies:*

$$\|(\theta(t), \mathbf{v}(t))\|_{H^{\varepsilon}(\Omega) \times \mathbf{H}^{\varepsilon}(\Omega)} \leq C e^{-\omega t},$$

*where* $C$ *depends on* $\|\theta_0\|_{H^{\varepsilon}(\Omega)}$ *and* $\|\mathbf{v}_0\|_{\mathbf{H}^{\varepsilon}(\Omega)}$.

# Numerical data

The heat source function

$$\phi_s(x, y) = 7\sin(2\pi x)\cos(2\pi y)$$

The controlled velocity and temperature profiles in the inflow boundary $\Gamma_{in}$

$$(\mathbf{u}_c, \tau_c) = \sum_{j=1}^{2} f_j(\mathbf{g}_v^j, g_\theta^j),$$

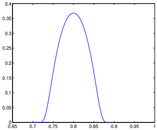
with

$$(\mathbf{g}_v^1, g_\theta^1) = ((\alpha(y), 0), 0), \quad (\mathbf{g}_v^2, g_\theta^2) = ((0, 0), \beta(y)).$$

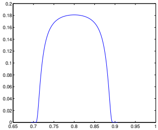
The quantities $(f_1, f_2)$ are the control variables to be calculated and we take

$$\alpha(y) = \exp\left(-\frac{0.0001}{[(0.7 - y)(0.9 - y)]^2}\right), \quad \beta(y) = 0.2\alpha(y)$$

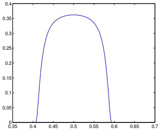
# Control functions $g$




(a) $b_{\mathbf{v}}(y) = e^{-\frac{0.0001}{[(0.7-y)(0.9-y)]^2}}$

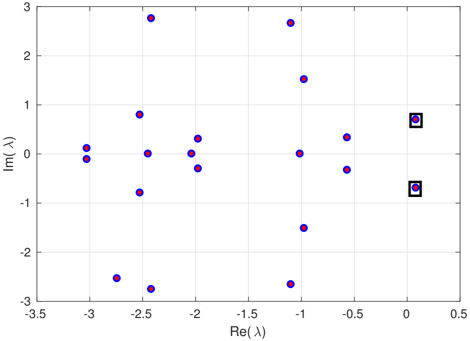



(b) $b_{\theta_{\mathrm{I}}}(y) = 0.2e^{-\frac{0.00001}{[(0.7-y)(0.9-y)]^2}}$

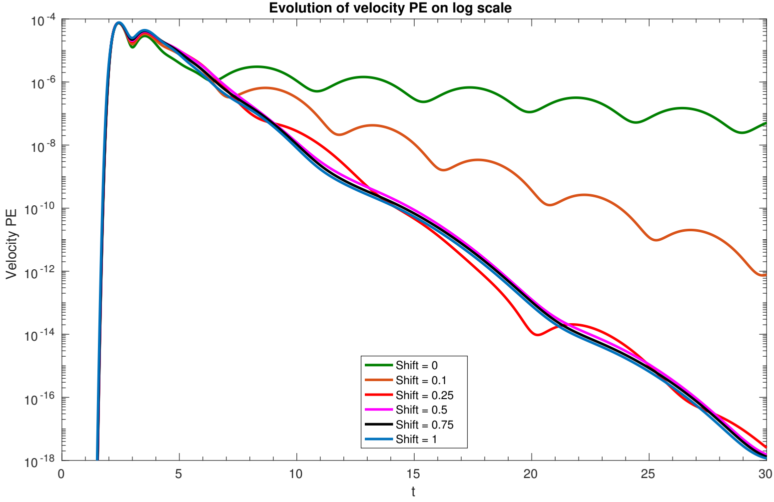



(c) $b_{\theta_{\mathrm{H}}}(x) = 0.4e^{-\frac{0.00001}{[(0.4-x)(0.6-x)]^2}}$

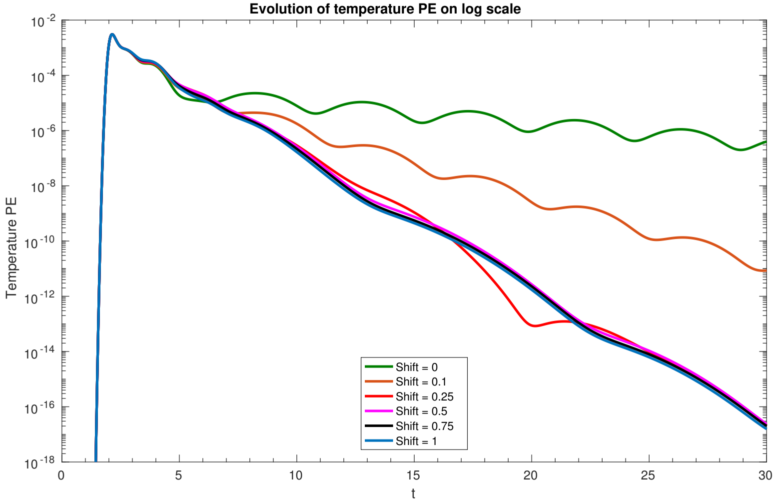
# Eigenvalues of the linearized Boussinesq operator at Reynold's number $100$



# Evolution of velocity perturbation energies on $\log_{10}$ scale



# Evolution of temperature perturbation energies on $\log_{10}$ scale



# Conclusions

- Linearized the system around a steady state with mixed boundary conditions,
- Determined the loss of regularity of the solutions.
- Using suitable weighted Sobolev spaces, proved Green's formula and calculated adjoint operators
- Proved stabilizability of the linearized system by showing Hautus condition holds
- Computed a finite dimensional feedback boundary control by solving a finite dimensional algebraic Riccati equation
- Using this study, proved local stabilization of the nonlinear system
- Numerically implemented the procedure to verify exponential stabilization

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All our best wishes Jean-Pierre!