Optimal Control Problems and Homogenization

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Application to Calculus of Variations:

Optimal Design Problem (ODP):

Control : Characteristic function: χ_{ω_A}

State equation:

$$u_{\omega_A} \in H_0^1(\Omega)$$

$$-\text{ div }(A_{\omega_A} \nabla u_{w_A}) = g \in H^{-1}(\Omega)$$

$$\text{where } A_{\omega_A} = a_1 \chi_{\omega_A} + a_2(1 - \chi_{\omega_A}).$$

Cost functional :

$$J(\chi_{\omega_A}) := \int_\Omega A_{\omega_A}(x) \nabla u_{\omega_A} \cdot \nabla u_{\omega_A} dx = \int_\Omega g u_{\omega_A} dx.$$

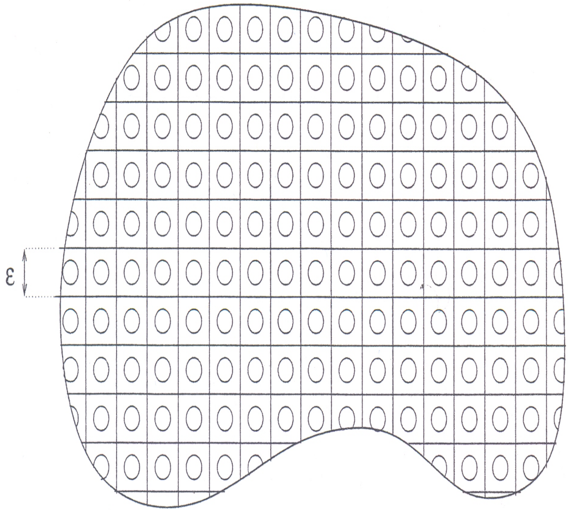
Minimization problem:

$$\inf \{ J(\chi_{\omega_A}); |\omega_A| = \delta_A |\Omega| \}$$

Examples of complex media/microstructures (Mixing process): Laminates, Periodic structures, Hashin-Shtrikman structures, etc. with two phases:

$$A^\epsilon(x) = a_1\chi_{\omega_{A^\epsilon}}(x) + a_2\left(1 - \chi_{\omega_{A^\epsilon}}(x)\right)$$

In pictures, we can easily see the set ω_{A^ϵ} and its complement.



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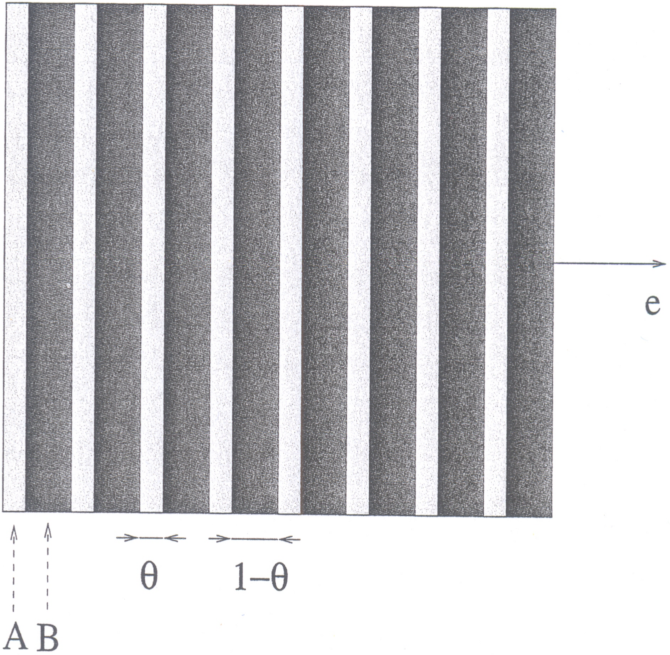
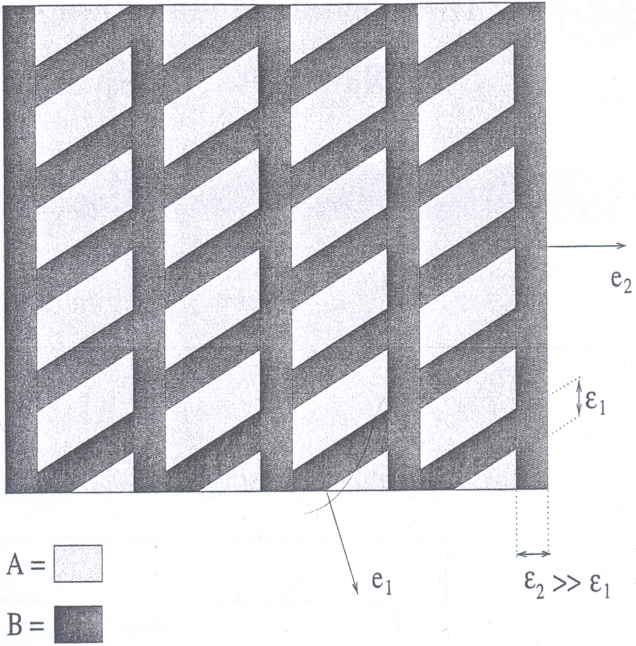
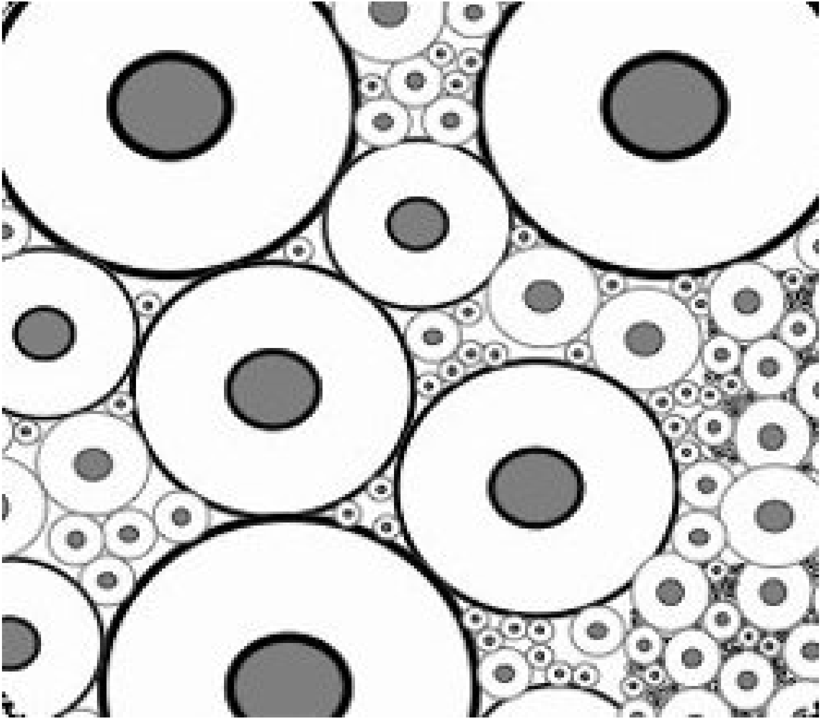

e
θ
1−θ
A
B



e2
ε1
e1
ε2 >> ε1
A =
B =



Minimization over characteristic functions.
Difficulties:Non-existence of minimizers,
lack of convexity,
not-so-rich variations among characteristic functions to deduce optimality conditions. (e-g:Hadamard variations).
Remedy:Relaxation.
Roughly speaking, add limit points of all minimizing sequences.
In our model ODP, this requires H-limits and the resolution of the G-closure problem.
Fortunately, both issues are solved for two-phase media.
Minimizers to ODP are found among $N-$rank laminates.

RELAXATION.

$$\inf\{F_0(x_0); x_0 \in X_0\} \to \inf\{F(x); \quad x \in X\}$$

Conditions on the Relaxed Problem:

(a) X is compact

(b) $F : X \to R$ is lower semi continuous

(c) X_0 is dense in X

(d) $F\,|_{X_0} = F_0$

(e) For each $x \in X$, there exists $\{x_0^{(n)}\} \subset X_0$ such that

$$x_0^{(n)} \to x \text{ in } X, \liminf \quad F_0(x_0^{(n)}) = F(x)$$

Consequences. Relaxed problem has at least one solution. Limit point of any minimizing sequence is a solution. Further, minimum values are the same. Solution of the original problem, if any, is a solution to the relaxed problem.

Local volume proportion of components $\{a_1, a_2\}$ in a mixture is $\{\theta_A, 1-\theta_A\}$ where

$$\omega_{A^\epsilon}(x) \rightharpoonup \theta_A(x) \text{ in } L^\infty(\Omega) \text{ weak*}$$

$$\omega_A^\epsilon \to \chi_{\omega_A^\epsilon}$$

Notion of weak limit is adequate.

Heat Conductivity of a mixture:
$A^\epsilon \rightharpoonup \overline{A}$ in $L^\infty(\Omega)$ weak*
This weak limit is not appropriate.

We need the notion of **Homogenization, *H*-convergence, *H*-limit:**
We say that A^ϵ is $H-$convergent to $A^\star$

$$A^\epsilon \quad \stackrel{H}{\rightharpoonup} \quad A^\star \text{ if}$$

$$A^\epsilon \nabla u^\epsilon \quad \rightharpoonup \quad A^\star \nabla u \text{ in } L^2(\Omega) \text{ weak,}$$

for all test sequences $\{u^\epsilon\}$ such that

$$\begin{array}{rcl} u^\epsilon & \rightharpoonup & u \text{ weakly in } L^2(\Omega), \\ \nabla u^\epsilon & \rightharpoonup & \nabla u \text{ weakly in } L^2(\Omega), \\ \text{div } (A^\epsilon \nabla u^\epsilon) & & \text{is strongly convergent in } H^{-1}(\Omega). \end{array}$$

(Oscillating system with differential constraints).

It follows that not only we have convergence of heat flux, but also canonical energy densities converge:

$$A^\epsilon \nabla u^\epsilon \rightharpoonup A^\star \nabla u \text{ in } L^2(\Omega)$$

$$A^\epsilon \nabla u^\epsilon \cdot \nabla u^\epsilon \rightharpoonup A^\star \nabla u \cdot \nabla u \text{ in } D'(\Omega).$$

About A^*:

A^* is a macro quantity depending on conductivities of individual components of the mixture,their volume proportions and more importantly on the microstructure. We are interested in getting estimates on A^* independent of microstructures,using only macro quantities.
A^* is local quantity.

First Task: Compactness Theorem showing the existence of mixtures:
Theorem: Given A^ϵ, there is a subsequence which H-converges to some A^* which represents the mixture.

Compactness means some sort of stability. Secondly, H-limit of Fourier materials is again Fourier. This means that the mixture is a Fourier material. However, H-limit of isotropic materials need not be isotropic.
These results provide confirmation of correctness of the topology for the mixing process.

Second Task: Initial engineering problem is Optimal Design according to specific criterion (modeled by ODP).Mathematicians aimed at more ambitious goal of characterizing all possible designs (called **G-closure problem**) and then ODP resolution amounts to making a choice among them. The surprise is that the harder problem of G-closure can be solved in certain cases.

Theorem (Murat-Tartar, Lurie-Cherkaev): Conductivities of various mixtures that can be formed by varying microstructures are characterized by a set of **bounds/inequalities** in the space of conductivities. These define a convex lens shaped region. They are **optimal** in the whole physical domain. Theorem also constructs the microstructures/ mixing process corresponding to these bounds. In other words, the result describes full details of all possible materials obtained.

Once we have such a result, it is then easy to choose the desired material behaviour via optimization.

Celebrated Theorem (Murat-Tartar, Lurie-Cherkaev)
H-limits $A^\star$ are charcterized by the following inequalities:

$$\underline{A}(x) \leq A^*(x) \leq \overline{A}(x),$$

$$Tr\{(A^* - a_1 I)^{-1}\} \quad \leq \quad \frac{1}{N} Tr\{(\underline{A} - a_1 I)^{-1}\} + \frac{N-1}{N} Tr\{(\overline{A} - a_1 I)^{-1}\}$$

$$Tr\{(a_2 I - A^*)^{-1}\} \quad \leq \quad \frac{1}{N} Tr\{(a_2 I - \underline{A})^{-1}\} + \frac{N-1}{N} Tr\{(a_2 - \overline{A})^{-1}\}$$

Here $\overline{A}$ and $\underline{A}$ are respectively arithmetic and harmonic means of $\{A^\varepsilon\}$:

$$A^\varepsilon \quad \rightharpoonup \quad \overline{A} \text{ in } L^\infty(\Omega) \text{ weak } *$$

$$(A^\varepsilon)^{-1} \quad \rightharpoonup \quad (\underline{A})^{-1} \text{ in } L^\infty(\Omega) \text{ weak } *$$

Above bounds are in terms of eigenvalues of $A^\star$ which represent conductivity of the mixture in the eigenvector directions. They are **optimal** in the following sense : Any $A^\star$ satisfying above bounds is a H-limit of two-phase mixtures with some microstructures.

Phase diagram in the space of macro parameters $A^\star$ is depicted in the following picture: convex lens shaped region.

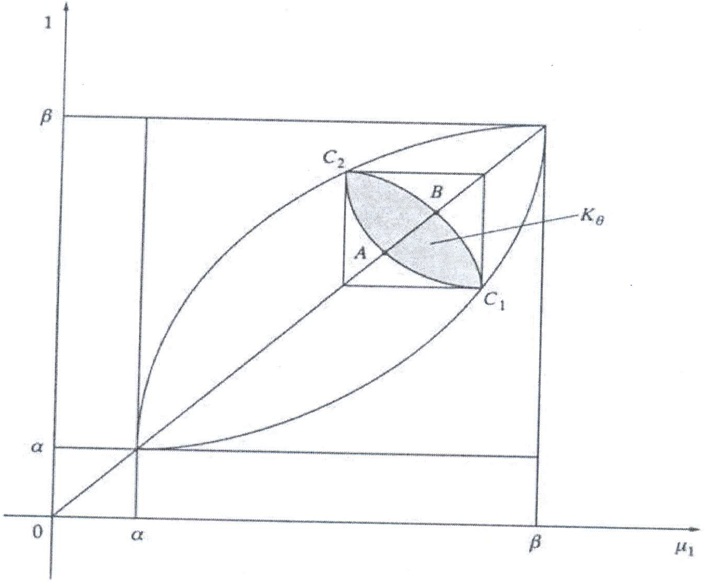


Figure 1.

Mathematical ideas behind proving these results:
Div-Curl Lemma, Compensated Compactness,H-measures, Wave cone etc.

At a fundamental level, the central problem of Homogenization is to find the (weak) limit of quadratic quantities of oscillating sequences satisfying certain differential constraints.In fact, the very definition of the homogenized matrix is based on the following framework:
Gradient of the temperature field oscillate satisfying differential constraint defined by balance equations.Also curl-free condition. The corresponding energy density is a quadratic quantity incorporating the interaction between the heat flux and the gradient of the temperature field. We are interested in the (weak) limit of this quantity and the homogenized matrix appears in the limiting energy density.

Thus we consider an **oscillating sequence** (with zero mean) satisfying certain differential constraints.We view them as waves.They have well-defined **location** in the physical space and well-defined **directions** of propagation and also well-defined **amplitudes**. Such bad amplitudes constitute the **wave cone** of the system. Squaring these amplitudes, we get the **intensity** of these waves.

The significance is that using the above ingredients, we can constuct waves whose presence is an obstruction to the compactness (strong convergence, stability) of the sequence.

H-measure, denoted by μ is a measure on the phase space,associated to such an oscillating sequence.It is a weak limit constructed from all the above pointwise ingredients. It quantifies the non-compactness of the given oscillating sequence.Indeed, if μ is zero, then the sequence is compact (and conversely).

Relaxed ODP

Control : (θ_A, A^*).

$$0 \leq \theta_A(x) \leq 1, \frac{1}{|\Omega|} \int\limits_{\Omega} \theta_A = \delta_A$$

$$A^*(x) \in M_{\theta(x)} x \in \Omega$$

State Equation.

$$U_{\theta_A} \in H_0^1(\Omega)$$

$$-\text{ div } (A^* \nabla u_{\theta_A}) = g \text{ in } \Omega$$

Cost functional

$$J(\theta_A, A^*) = \int\limits_{\Omega} g u_{\theta_A}$$

Minimization Problem

$$\min \{ J(\theta_A, A^*); \theta_A, A^* \}$$

Numerics of ODP: Exact optimum solution is of course a genuine microstructure/composite. But in practice, we need a two-phase material. (one of them is void in case of shape optimization). Since ODP is a relaxed version of the corresponding ODP with classical materials, there is one classical microstructure lying near (w.r.t H-**topology**) our optimal relaxed microstructure for which the objective functional value is close to the optimal one.There are many techniques to implement this idea in Numerics.

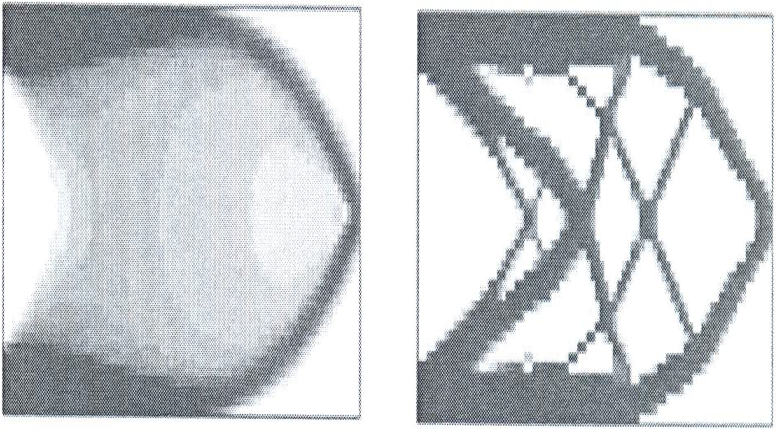


Figure 5.8: Optimal shape of the cantilever: composite (left) and penalized (right).

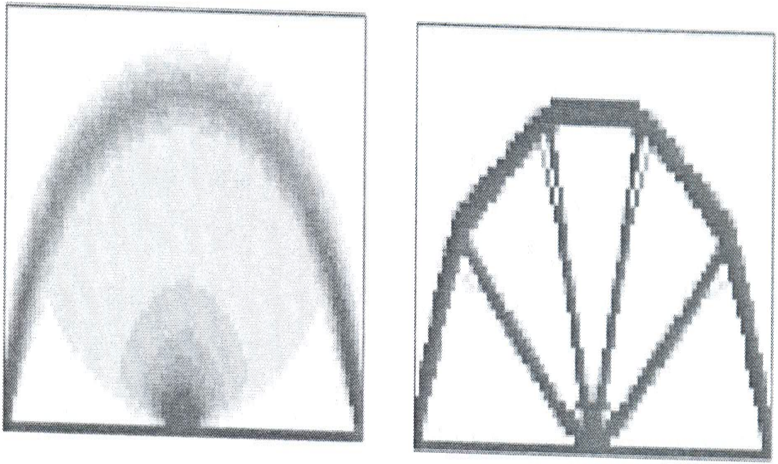


Figure 5.11: Optimal shape of the bridge: composite (left) and penalized (right).

Optimal Oscillation-Dissipation Problem (OODP):
Controls : Two characteristic functions: $(\chi_{\omega_A}, \chi_{\omega_B})$
State equation:

$$u_{\omega_A} \in H_0^1(\Omega)$$

$$-\text{ div } (A_{\omega_A} \nabla u_{w_A}) = g \in H^{-1}(\Omega)$$

where $A_{\omega_A} = a_1 \chi_{\omega_A} + a_2(1 - \chi_{\omega_A})$ (with two phases).

Cost functional :

$$J(\chi_{\omega_A}, \chi_{\omega_B}) := \int_\Omega B_{\omega_B}(x) \nabla u_{\omega_A} \cdot \nabla u_{\omega_A} dx.$$

with $B_{\omega_B} = b_1 \chi_{\omega_B} + b_2(1 - \chi_{\omega_B})$ (with two phases).
Minimization problem:

$$\inf \left\{ J(\chi_{\omega_A}, \chi_{\omega_B}); |\omega_A| = \delta_A |\Omega|, |\omega_B| = \delta_B |\Omega| \right\}.$$

Interpretation of Solution to OODP:

Basic idea is that the total energy of the oscillations after appropriate dissipation in the whole domain is minimized.This is captured by minimization wrt both (A, B) of the cost/objective functional.

Let $(A^*, B^\#)$ be optimal solution.

The minimizer $(A^*, B^\#)$ is a macro representation of Oscillation-Dissipation processes which co-exist in the system.

Why OODP is not trivial?

Ideally speaking, we would like to put minimum value of B in regions of large gradient of the state u .The difficulty is that such regions are not known apriori. It is part of the minimization problem. Moreover, these regions can be large whereas minimum value material of B can have small volume.
Secondly, there is an interaction between microstructures ω_A, ω_B.
Thirdly, when A =B, the OODP coincides with ODP and so non-trivial.

As before, there are four tasks in the resolution of OODP.

1) New Notion of Convergence is required: Notion of convergence relative to a microstructure:
2) Compactness Theorem
3) Optimal Bounds on emerging new Macro Parameters.
4) Computation of macro parameters on appropriate microstructures to show all points of the phase diagram can be reached via H-convergence & relative convergence.

Main feature of the problem which is present here but not in previous ODP:Interaction between microstructures.

Notion of Convergence relative to a microstructure:

A^ϵ, B^ϵ are given.
We say $\{B^\epsilon\}$ converges to $B^\#$ relative to A^ϵ if

$$B^\epsilon \nabla u^\epsilon \cdot \nabla u^\epsilon \rightharpoonup B^\# \nabla u \cdot \nabla u \text{ in } D'(\Omega),$$

for all test sequences $\{u^\varepsilon\}$ such that

$$\begin{array}{rcl} u^\epsilon & \rightharpoonup & u \text{ in } H^1(\Omega) \text{ weak} \\ -\operatorname{div}(A^\epsilon \nabla u^\epsilon) & \to & H^{-1}(\Omega) \text{ strong.} \end{array}$$

Notation: $B^\epsilon \overset{A^\epsilon}{\rightharpoonup} B^\#$.
$B^\#$ is a new macro matrix apart from A^*.

Oscillating Test functions in Homogenization

$$\chi^\varepsilon \xrightarrow{L^2} \xi \cdot x$$

$$\nabla \chi^\varepsilon \xrightarrow{L^2} \xi \in \mathbb{R}^N$$

$$\text{div } (A^\varepsilon \nabla \chi^\varepsilon) \xrightarrow{H^{-1}}$$

Define $A^* \xi =$ weak limit of $A^\varepsilon \nabla \chi^\varepsilon$

Adjoint oscillating Test functions for relative limits

$$\psi^\varepsilon \overset{L^2}{\rightharpoonup} \psi$$

$$\nabla\psi^\varepsilon \overset{L^2}{\rightharpoonup} \nabla\psi$$

$$\text{div}\,(A^\varepsilon\nabla\psi^\varepsilon - B^\varepsilon\nabla\chi^\varepsilon) = 0$$

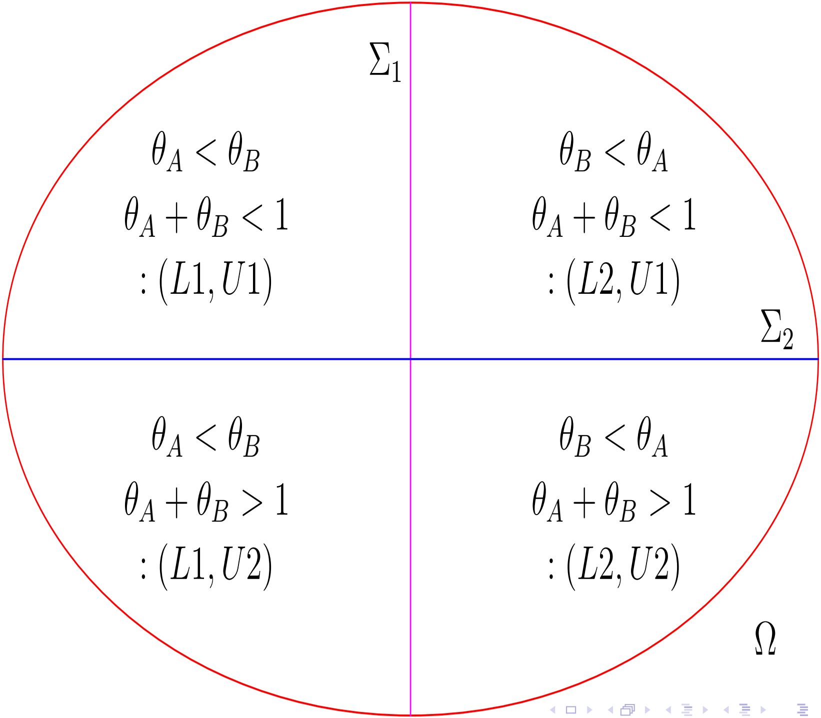
$$B^\# = A^*\psi - \text{ weaklimit }\{A^\varepsilon\nabla\psi^\varepsilon - B^\varepsilon\nabla\chi^\varepsilon\}$$

$B^{\#}$ is an outcome of interaction between microstructures $\{A^{\epsilon}, B^{\epsilon}\}$. The case $B^{\epsilon} = A^{\epsilon}$ is called **self-interacting case**. In this case, $B^{\#} = A^{\star}$ and so there are no new macro parameters. This case coincides with the previously treated case. Thus we are dealing with a genuine extension of the old problem. In the extended problem, there are new macro parameters $B^{\#}$ apart from $A^{\star}$.

The two lower bounds $(L1), (L2)$ reduce to the two known bounds for A^{*}. The two upper bounds $(U1), (U2)$ define certain regions in the phase space of A^{*} which are not of any importance.

New Results:

The original physical domain Omega is divided into four sub-domains with interfaces between them. There are four optimal regions in the phase space of macro parameters $(A^{\star}, B^{\#})$, corresponding to four sub-domains.(See Picture). These four regions are defined by four inequalities labeled as $\{L1, L2, U1, U2\}$. Because of these structural changes, minimizers for OODP are found among N-rank laminates with an interface across which core-matrix values get switched.
In the classical case, there was only one region and consequently, ODP solutions did not have interfaces.


Σ_1
$\theta_A < \theta_B$
$\theta_A + \theta_B < 1$
$: (L1, U1)$
$\theta_B < \theta_A$
$\theta_A + \theta_B < 1$
$: (L2, U1)$
Σ_2
$\theta_A < \theta_B$
$\theta_A + \theta_B > 1$
$: (L1, U2)$
$\theta_B < \theta_A$
$\theta_A + \theta_B > 1$
$: (L2, U2)$
Ω

Lower Trace Bound (L1)

There exists a unique $\theta(x) \leq \theta_A(x)$ such that

$$Tr(A^\star(x) - a_1 I)^{-1} = Tr(\overline{A}_\theta(x) - a_1 I)^{-1} + \frac{\theta(x)}{(1-\theta(x))a_1}$$

where $\overline{A}_\theta(x) = \{a_1\theta(x) + a_2(1-\theta(x))\} I$.
Then we have

$$Tr\left\{(B^\#(x) - b_1 I)(\overline{A}_\theta(x) - a_1 I)^2 (A^\star(x) - a_1 I)^{-2}\right\}$$

$$\geq N(b_2 - b_1)(1 - \theta_B(x) + \frac{b_1(a_2 - a_1)^2}{a_1^2}\theta(x)(1-\theta(x)).$$

This is optimal in the sub-domain $\{x : \theta_A(x) \leq \theta_B(x)\}$.

Upper Trace Bound (U1)

$$Tr\left\{\left(\frac{b_2}{a_1}A^*(x) - B^{\#}(x)\right)(\overline{A}_\theta(x) - a_1 I)^2(A^*(x) - a_1 I)^{-2}\right\}$$
$$\geq N(b_2 - b_1)\theta_B(x) + N\frac{b_2}{a_1}(a_2 - a_1)(1 - \theta(x)).$$

This is optimal in the sub-domain $\{x : \theta_A(x) + \theta_B(x) \leq 1\}$.

For arbitrary structures, *H*-**measure** is the right tool. **Differential Relations** with variable coefficients expressing compactness (so crucial in the theory) can be equivalently transformed to **algebraic relations** involving H-measures. In a sharp sophisticated way, the above algebraic relation ensures the decay of short waves in the system and thereby implies compactness. Such a sharp result is needed to study interaction between two microstructures.

Remarks:

1. Shift the focus from Material Science to PDE $A*$ can be seen as a macro quantity parametrizing all weak limits of the oscillating field (solution of PDE) as microstructure (variable coefficients) varies. Similarly, $B\#$ is a macro quantity parametrizing all weak limits of energy density of the oscillating field. For two-phased media, characterization of $A*$ is complete whereas characterization of $B\#$ is not complete because we allow only special test matrices B. Note that $A*$ is a particular value of $B\#$, but set of $B\#$values is much larger showing that oscillations of energy density is much richer. This richness shows many more possible solutions to OCP, whose objective functional is based on energy density.

2. In Homogeniztion Theory, one is interested in the weak limits of quadratic functions. That is why, Compensated Compactness, Defect measures, H-measures were introduced.But rough coefficients pose a challenge. In the particular case of elliptic equation, the difficulty is overcome.

3. Correctors in Homogenization: Idea is to obtain strong convergence of the oscillating field by subtracting oscillations.In doing so, there are hypotheses on underlying microstructures.There is no optimal result for general microstructures. Our approach does not seek strong convergence but weak limits and in doing so, we find new macro parameters $B\#$. Thanks to relative limits, we can say strong convergence iff $(Id)\# = Id$.

Remarks:

4. Minimization wrt B ensures minimum value of B at large values of $|\text{ grad } u|^2$ and large value of B at small values of $|\text{ grad } u|^2$ Note large values of $|\text{ grad } u|^2$ occur only at interface between the two-phases of A. Therefore minimizer $B\#$ tries to capture this region (called Thermoclines). This gives potentially additional information / observation / measurement about the optimal A-microstructure which is not available in (ODP). In principle, this macro information is useful in Inverse Problems involving microstructures. In this context, recall that different microstructures may give rise to the same $A*$, but have different $B\#$ for some B-microstructure.

5.In OCP, State equation and observations are different entities and consequently the natural convergences for them are not the same, but related. Natural topology for observation must be adapted to that of the State. The same issue was also encountered by Lions in his attempt to homogenize exact controllability problem for the Wave Equation. However, he had additional difficulties because the system was hyperbolic.

Remarks:

6. (OODP) reminds one of Phase Change Material (PCM) which is used in latent heat energy storage and release, by means of change in microstructure. Substance remains the same but there is a phase change at the same temperature. This corresponds to $\{A, B\}$ with same values of conductivities, but with different microstructures. Maximizing wrt B would amount to maximizing latent heat energy by means of microstructures. Homogenized (macro) temperature remains the same during B-optimization.

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a Jean-Pierre
toutes nos felicitations.
Merci infiniement.

எங்கள் வாழ்த்துக்கள், எங்கள் பாராட்டுக்கள்