

# Sensitivity relations of the costate variable for a class of state-constrained control problems

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- Let  $T > 0$  be a given time horizon. For a given non-empty compact subset  $U$  of  $\mathbb{R}^m$  ( $m \geq 1$ ), define the set of admissible controls as:

$$\mathcal{U} := \left\{ u : (0, T) \rightarrow \mathbb{R}^m, \text{ measurable, } u(t) \in U \text{ a.e.} \right\}.$$

- Consider the following control system:

$$(1) \quad \begin{cases} \dot{y}(s) := f(y(s), u(s)), & \text{a.e. } s \in [t, T] \\ y(t) := x, \end{cases}$$

where  $f$  is continuous on  $\mathbb{R}^d \times U$  into  $\mathbb{R}^d$ , it is loc. Lipschitz continuous w.r.t  $y$ , and there exists  $C > 0$  s.t.  $|f(y, u)| \leq C(1 + |y|)$  for all  $y \in \mathbb{R}^d$ ;

- For every  $t \in [0, T]$ , we define the set of trajectories:

$$\mathcal{S}_{[t, T]}(x) := \left\{ y_{t,x} \in W^{1,\infty}([t, T]; \mathbb{R}^d), y_{t,x} \text{ satisfies (1) for some } u \in \mathcal{U} \right\}.$$

The multi-application  $(t, x) \rightsquigarrow \mathcal{S}_{[t, T]}(x)$  is **Lipschitz continuous**.

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Moreover, for every  $y \in \mathbb{R}^d$ ,  $f(y, U)$  is a convex set;

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The multi-application  $(t, x) \rightsquigarrow \mathcal{S}_{[t, T]}(x)$  is **Lipschitz continuous**.

The set of trajectories  $\mathcal{S}_{[t, T]}(x)$  is a compact subset of  $C^0([t, T]; \mathbb{R}^d)$ .

## Endpoint constraint

$$\begin{aligned} \min \quad & \Phi(y(T)) \\ \text{s.t.} \quad & y \in \mathcal{S}_{[t, T]}(x), \quad g_0(y(T)) \leq 0. \end{aligned}$$

- ▶  $g_0 : \mathbb{R}^d \rightarrow \mathbb{R}$  is a Lipschitz continuous function ;
- ▶ The final cost  $\Phi : \mathbb{R}^d \rightarrow \mathbb{R}$  is Lipschitz continuous.

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## Intermediate constraints

$$\begin{aligned} \min \quad & \Phi(y(T)) \\ \text{s.t.} \quad & y \in \mathcal{S}_{[t, T]}(x), \\ & g_1(y(t_1)) \leq 0, \quad g_0(y(T)) \leq 0. \end{aligned}$$

- ▶  $g_0$  and  $g_1$  are Lipschitz continuous functions from  $\mathbb{R}^d$  into  $\mathbb{R}$ ; and  $t_1$  is in  $]0, T[$ ;
- ▶ The final cost  $\Phi : \mathbb{R}^d \rightarrow \mathbb{R}$  is Lipschitz continuous.

# Outline

- 1 Mayer problem without state constraints
- 2 Endpoint constrained control problems
- 3 Control problems with intermediate constraints

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## Sensitivity analysis - Pontryagin's Maximum Principle (PMP)

Mayer Problem (free of state constraints)

$$\begin{aligned} \min \quad & \Phi(y(T)) \\ \text{s.t.} \quad & \begin{cases} \dot{y}(s) := f(y(s), u(s)), & \text{a.e. } s \in [0, T], \\ y(0) := x_0, \\ u(s) \in U \text{ a.e. } [0, T]. \end{cases} \end{aligned}$$



# Sensitivity analysis - Pontryagin's Maximum Principle (PMP)

## Mayer Problem without state constraints

► Let  $(\bar{y}, \bar{u})$  an optimal pair. There exists  $\bar{p} \in W^{1,\infty}([0, T]; \mathbb{R}^d)$  satisfying:

$$\dot{\bar{y}}(s) = f(\bar{y}(s), \bar{u}(s)), \quad \bar{y}(0) = x_0,$$

$$-\dot{\bar{p}}(s) = [D_y f(\bar{y}(s), \bar{u}(s))]^T \bar{p}(s), \quad \bar{p}(T) = \nabla \Phi(\bar{y}(T)),$$

$$H(\bar{y}(s), \bar{p}(s), \bar{u}(s)) = \max_{u \in U} H(\bar{y}(s), \bar{p}(s), u), \quad \text{a.e. } s \in [0, T].$$

► where  $H(x, p, u) := -\langle p, f(x, u) \rangle$

Ref: Boltyanskii'57, Pontryagin et al.'61-62, Warga'72, Sussman'94-98, Vinter'00, Clarke'13, Mordukhovich'06 ...

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## Sensitivity analysis - Hamilton-Jacobi-Bellman (HJB) approach

$$\vartheta(t, x) = \min_{y \in \mathcal{S}_{[t, T]}(x)} \Phi(y(T))$$

$$\begin{aligned}\vartheta(t, x) &= \min_{y \in \mathcal{S}_{[t, t+h]}} \vartheta(t+h, y(t+h)) \quad h \in (0, T-t). \\ \vartheta(T, x) &= \Phi(x)\end{aligned}$$

► **Suboptimality:**

$$\forall y \in \mathcal{S}_{[t, T]}(x), \quad \vartheta(t, x) \leq \vartheta(t+s, y(t+s)) \quad \text{for } s \in (0, T-t],$$

► **Superoptimality**

$$\exists y \in \mathcal{S}_{[t, T]}(x), \quad s \mapsto \vartheta(t+s, y(t+s)) \text{ is constant}$$

- $\vartheta$  is the unique bounded lsc (or continuous) *viscosity* solution of the HJB equation:

$$\begin{aligned} -\partial_t \vartheta(t, x) + \max_{u \in U} (-D_x \vartheta(t, x) \cdot f(x, u)) &= 0 \\ \vartheta(T, x) &= \Phi(x). \end{aligned}$$

Case  $\Phi$  Lipschitz continuous: Bellman, Subbotina, Evans, P.L Lions, Cappuzo-Dolcetta/Bardi, Barles,...

Case  $\Phi$  lsc: Barron-Jensen, Frankowska, ...

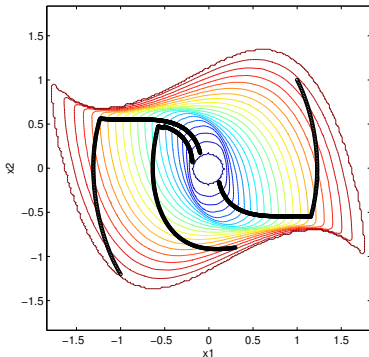
➤ Van der Pol dynamics :

$$\begin{cases} \dot{y}_1(s) = y_2, \\ \dot{y}_2(s) = -y_1 + y_2(1 - y_1^2) + u(s), \\ y(t) = x, \\ u(s) \in [-1, 1]. \end{cases}$$

➤ The cost function:

$$\Phi(y) = d_C(y),$$

where  $C := \{y, |y| \leq r_0\}$ .



➤ The HJB equation to be solved for this problem is:

$$-\partial_t \vartheta(t, x) + \begin{pmatrix} -x_2 \\ x_1 + x_2(1 - x_1^2) \end{pmatrix} \cdot D_x \vartheta(t, x) + |\partial_{x_2} \vartheta(t, x)| = 0$$

$$\vartheta(T, x) = d_C(x).$$

## Mayer Problem (free of state constraints)

$$\vartheta(t, x) := \min\{\Phi(y(T)) \mid y \in \mathcal{S}_{[t, T]}(x)\}.$$

for every  $x \in \mathbb{R}^d$  and  $T \geq 0$ .

- There exists  $\bar{p} \in W^{1, \infty}([0, T]; \mathbb{R}^d)$  satisfying:

$$\begin{aligned} -\dot{\bar{p}}(s) &= [f'_y(\bar{y}(s), \bar{u}(s))]^\top \bar{p}(s), \\ \bar{p}(T) &= \nabla \Phi(\bar{y}(T)). \end{aligned}$$

For a.e  $s \in [0, T]$ , we have:

$$H(\bar{y}(s), \bar{p}(s), \bar{u}(s)) = \max_{u \in U} H(\bar{y}(s), \bar{p}(s), u) \quad (=: \mathcal{H}(\bar{y}(s), \bar{p}(s))).$$

- Moreover,

$$\bar{p}(0) \in \partial_y \vartheta(0, x) \quad \& \quad \bar{p}(s) \in \partial_y \vartheta(s, \bar{y}(s)) \quad \text{a.e. on } [0, T]$$

$$(\mathcal{H}(\bar{y}(s), \bar{p}(s)), \bar{p}(s)) \in \partial_{s, y} \vartheta(s, \bar{y}(s)) \quad \text{a.e. on } [0, T].$$

Here,  $\partial_\xi$  stands for the partial generalized gradient w.r.t  $\xi$ .

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## Example:

$$\begin{array}{ll} \text{Minimize} & \Phi(y(1)) \\ \text{subject to} & \dot{y}(s) = y(s)u(s) \quad s \in (0, 1), \quad y(0) = 0, \\ & u(s) \in [0, 1] \quad s \in [0, 1]. \end{array}$$

Here, we consider

$$\Phi(x) := \begin{cases} -x & \text{if } x > 0, \\ -e^{1/2}x & \text{if } x \leq 0. \end{cases}$$

Then the value function is

$$\vartheta(s, y) = \begin{cases} -e^{1-s}y & \text{if } y > 0, \\ -e^{1/2}y & \text{if } y \leq 0. \end{cases}$$



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► The optimal trajectory is  $\bar{y} \equiv 0$  and an optimal control is  $\bar{u} \equiv 0$ .

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- The optimal trajectory is  $\bar{y} \equiv 0$  and an optimal control is  $\bar{u} \equiv 0$ .
- The costate satisfies  $-\dot{\bar{p}}(s) = \bar{u}(s)$  a.e on  $(0, 1)$  and  $\bar{p}(1) = \lambda \in \partial\Phi(\bar{y}(1))$ .

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$$\vartheta(s, y) = \begin{cases} -e^{1-s}y & \text{if } y > 0, \\ -e^{1/2}y & \text{if } y \leq 0. \end{cases} \quad \partial_y\vartheta(s, 0) = [-e^{1/2}, -e^{1-s}]$$

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- ▶ The costate satisfies  $-\dot{\bar{p}}(s) = \bar{u}(s)$  a.e on  $(0, 1)$  and  $\bar{p}(1) = \lambda \in \partial\Phi(\bar{y}(1))$ .
- ▶ In this example, there are uncountable many costate functions.
- ▶ The sensitivity relation  $\bar{p}(s) \in \nabla_y\vartheta(s, \bar{y}(s))$  is satisfied only for  $\lambda = -e^{1/2}$ .

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- 1 Mayer problem without state constraints
- 2 Endpoint constrained control problems**
- 3 Control problems with intermediate constraints

- Consider the endpoint constrained control problem

$$\begin{aligned} \vartheta(t, x) &:= \min \Phi(y(T)) \\ \text{s.t.} \quad &y \in \mathcal{S}_{[t, T]}(x), \\ &g_0(y(T)) \leq 0. \end{aligned}$$

- In general, the value function  $\vartheta$  is merely l.s.c., and may take infinite values (even when  $\Phi$  and  $g_0$  are  $C^\infty$ !)

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- In general, the value function  $\vartheta$  is merely l.s.c., and may take infinite values (even when  $\Phi$  and  $g_0$  are  $C^\infty$ !)
- **If** an optimal solution  $(\bar{y}, \bar{u})$  exists, then the PMP holds with the adjoint state satisfying:

$$\begin{aligned} -\dot{\bar{p}}(s) &= H'_y(\bar{y}(s), \bar{p}(s), \bar{u}(s)), \\ \bar{p}(T) &\in \mu \nabla \Phi(\bar{y}(T)) + \lambda_0 \partial g_0(\bar{y}(T)) \end{aligned}$$

where  $(\mu, \lambda_0) \in \{0, 1\} \times \mathbb{R}^+$ ,  $(\mu, \lambda_0) \neq 0$  and  $\min(\lambda_0, -g_0(\bar{y}(T))) = 0$ .

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- At this stage, it is not clear how to generalize the link between the PMP and HJB since  $\vartheta$  is only l.s.c



## An auxiliary control problem (Altarovici-Bokanowski-HZ'13)

- Consider the following auxiliary control problem ( $z \in \mathbb{R}$ ):

$$w(t, x, z) := \inf_{y \in \mathcal{S}_{[t, T]}(x)} \left\{ (\Phi(y(T)) - z) \vee g_0(y(T)) \right\}$$

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- The function  $w$  is the unique continuous solution to:

$$-\partial_t w(t, x, z) + \sup_{u \in U} (-D_x w(t, x, z) \cdot f(x, u)) = 0 \quad \text{on } [0, T) \times \mathbb{R}^{d+1},$$

$$w(T, x, z) = (\Phi(x) - z) \vee g_0(x) \quad \text{on } \mathbb{R}^{d+1}.$$

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## Improvement function (Mifflin'77, Solodov-Sagastizábal'04, Apkarian-et-al.'08, ...)

$$(P) \quad \min_{G(X) \leq 0} F(X)$$

- The auxiliary optimization problem:

$$(\hat{P}) \quad \min_X \left\{ (F(X) - z) \vee G(X) \right\}$$

- We have the equivalence

$$\bar{X} \text{ is optimal for } (P) \Leftrightarrow \begin{cases} \bar{X} \text{ solution of } (\hat{P}) \text{ for } \bar{z} = F(\bar{X}), \\ \min_X \left\{ (F(X) - \bar{z}) \vee G(X) \right\} = 0. \end{cases}$$

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For every  $x \in \mathbb{R}^d$ , we have:

(i)  $\text{Epi } \vartheta(t, \cdot) = \left\{ (x, z) : w(t, x, z) \leq 0 \right\},$

(ii)  $\vartheta(t, x) = \min \left\{ z \in \mathbb{R}, w(t, x, z) \leq 0 \right\}$

(iii) Let  $\bar{y}$  be an optimal solution of the auxiliary control problem with  $\bar{z} = \vartheta(t, x)$ , then  $\bar{y}$  is also an optimal solution of the original endpoint constrained problem

## Relationship between the PMP and HJB (Bokanowski-Desilles-HZ'19)

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$$\begin{aligned} -\dot{\bar{p}}_x(s) &= H'_y(\bar{y}(s), \bar{p}_x(s), \bar{u}(s)) \\ -\dot{\bar{p}}_z(s) &= 0 \\ \begin{pmatrix} \bar{p}_x(T) \\ \bar{p}_z(T) \end{pmatrix} &\in \partial_{x,z} \left\{ (\Phi(\bar{y}(T)) - \bar{z}) \vee g_0(\bar{y}(T)) \right\} \end{aligned}$$

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where  $(\mu, \lambda_0) \in \{0, 1\} \times \mathbb{R}^+$ ,  $\mu + \lambda_0 = 1$ , and  $\min(\lambda_0, -g_0(\bar{y}(T))) = 0$ .

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where  $(\mu, \lambda_0) \in \{0, 1\} \times \mathbb{R}^+$ ,  $\mu + \lambda_0 = 1$ , and  $\min(\lambda_0, -g_0(\bar{y}(T))) = 0$ .

- ▶ For a.e  $s \in [0, T]$ ,

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## Relationship between the PMP and HJB (Bokanowski-Desilles-HZ'19)

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## A simplified problem

### Example (Goddard problem)

$$\dot{r}(s) = v(s)$$

$$\dot{v}(s) = \frac{1}{m} (F_T u(s) - F_D(r, v)) - \frac{1}{r(s)^2}$$

$$\dot{m}(s) = -\beta F_T u(s)$$

$$r(0) = 1, v(0) = 0, m(0) = 1$$

$r(\cdot)$  : altitude

$v(\cdot)$  : velocity

$m(\cdot)$  : masse

$u(\cdot)$  : % of the thrust

Here  $F_D(r, v) = 310 v^2 e^{-500(r-1)}$ ,  $\beta = 2$ , and  $F_T = 3.5$

- ▶ The ratio  $u(t)$  is subject to the following constraint:  $0 \leq u(t) \leq 1$ .
- ▶ The rocket's altitude satisfies the final constraint ( $r_f^* = 1.01$ ):  $r(T) \geq r^*$ .

The optimal control problem is the following (for  $T > 0$ ):

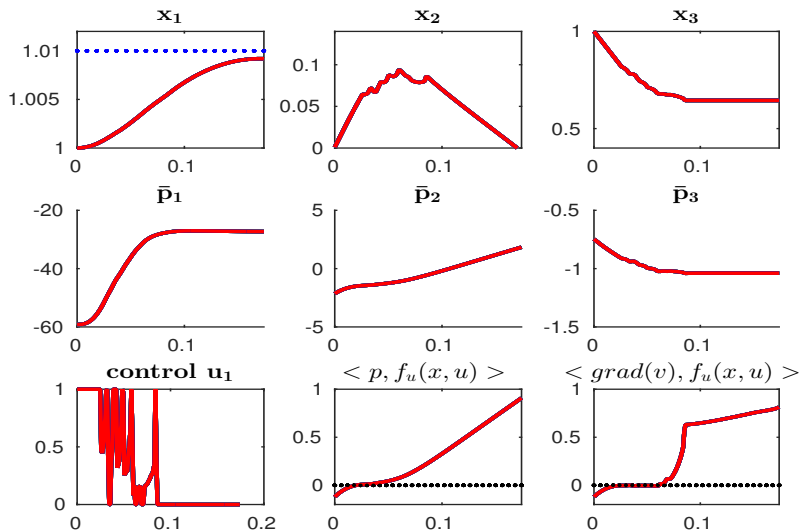
$$\left\| \begin{array}{l} \min -m(T) \\ u(t) \in [0, 1], \quad g_0(r(T)) := r^* - r(T) \leq 0. \end{array} \right.$$

- We first compute an approximation  $w^\Delta$  of the auxiliary value function  $w$
- Compute an approximation of  $\bar{p}^\Delta(0) = \nabla_x w^\Delta(0, x_0, \bar{z})$ .
- Solve the optimality condition with the value of  $\bar{p}^\Delta(0)$ . We get an optimal solution  $(y^\Delta, u^\Delta)$  of the control problem.
- We use the relations

$$\begin{aligned} p_x^\Delta(T) &= \mu \nabla \Phi(y^\Delta(T)) + \lambda_0 \nabla g_0(y^\Delta(T)) \\ -\mu &= \nabla_z w^\Delta(0, x_0, \bar{z}) \end{aligned}$$

and get approximate values for  $(\mu, \lambda_0)$ .

First approximation by using HJB equation on a coarse grid ( $T = 0.174$ )



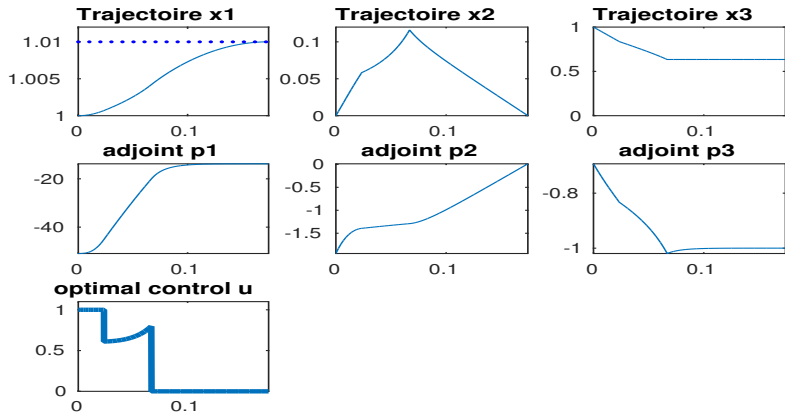


Figure: An optimal trajectory obtained by PMP principle combined with the HJB approach.

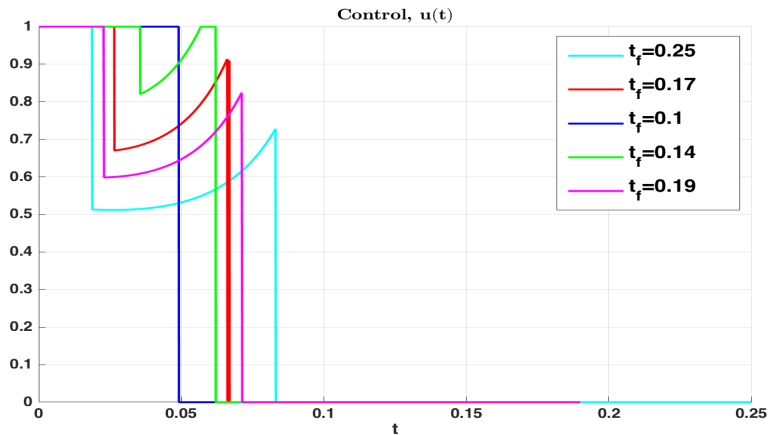


Figure: Optimal control laws for different final times.

# Outline

- 1 Mayer problem without state constraints
- 2 Endpoint constrained control problems
- 3 Control problems with intermediate constraints**



Intermediate constraints ( $t_1 \in ]0, T[$  is fixed)

$$\min \quad \Phi(y(T))$$

$$\text{s.t.} \quad y \in \mathcal{S}_{[t, T]}(x),$$

$$g_1(y(t_1)) \leq 0, g_0(y(T)) \leq 0.$$

- Consider the following auxiliary control problem ( $z \in \mathbb{R}$ ):

$$w(t, x, z) := \begin{cases} \inf_{y \in \mathcal{S}_{[t, T]}(x)} \{(\Phi(y(T)) - z) \vee g_0(y(T))\} & \text{for } t \in ]t_1, T] \\ \inf_{y \in \mathcal{S}_{[t, T]}(x)} \{(\Phi(y(T)) - z) \vee g_0(y(T)) \vee g_1(y(t_1))\} & \text{for } t \in [t_1, T] \end{cases}$$

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- The function  $w$  is continuous on  $[0, t_1[ \times \mathbb{R}^{d+1}$  and on  $]t_1, T] \times \mathbb{R}^{d+1}$ . It is the unique solution of:

$$\begin{aligned} -\partial_t w(t, x, z) + \mathcal{H}(x, D_x w(t, x, z)) &= 0 && \text{on } ]t_1, T] \times \mathbb{R}^{d+1}, \\ -\partial_t w(t, x, z) + \mathcal{H}(x, D_x w(t, x, z)) &= 0 && \text{on } [0, t_1[ \times \mathbb{R}^{d+1}, \\ w(T, x, z) &= (\Phi(x) - z) \vee g_0(x) && \text{on } \mathbb{R}^{d+1}, \\ w(t_1^-, x, z) &= w(t_1^+, x, z) \vee g_1(x) && \text{on } \mathbb{R}^{d+1}. \end{aligned}$$

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- ▶ Let  $\bar{y}$  be an optimal solution of the auxiliary control problem with  $\bar{z} = \vartheta(t, x)$ .

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$$-\dot{\bar{p}}_x(s) \in H'_y(\bar{y}(s), \bar{p}_x(s), \bar{u}(s))$$

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where  $(\mu, \lambda_0, \lambda_1) \in \{0, 1\} \times \mathbb{R}^+ \times \mathbb{R}^+$ ,  $\mu + \lambda_0 + \lambda_1 = 1$ ,  
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- ▶ Moreover, on  $]0, t_1[ \cup ]t_1^+, T[$ , we have:

$$\bar{p}_x(0) \in \partial_y w(0, x, \bar{z}), \quad \bar{p}_x(T) \in \partial_y w(T, \bar{y}(T), \bar{z}),$$

$$\bar{p}_x(t_1^\pm) \in \partial_y w(t_1^\pm, \bar{y}(t_1^\pm), \bar{z}),$$

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and  $(\mathcal{H}(\bar{y}(s), \bar{p}_x(s)), -\bar{p}(s)) \in \partial_{s,y,z} w(s, \bar{y}(s), \bar{z})$



## A very simple Example

$$\begin{aligned} \min \quad & \int_0^T y(s) ds \\ \text{s.t.} \quad & \dot{y}(s) = u(s), \\ & y(0) = 0, \quad y(T) \geq 1, \quad y(T/2) \geq 0.5, \\ & u(s) \in [-1, 1]. \end{aligned}$$

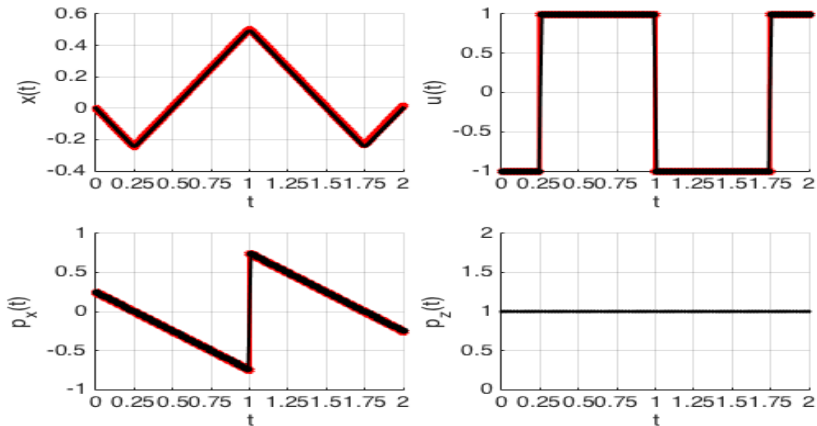


Figure: Case where  $T = 2$ .

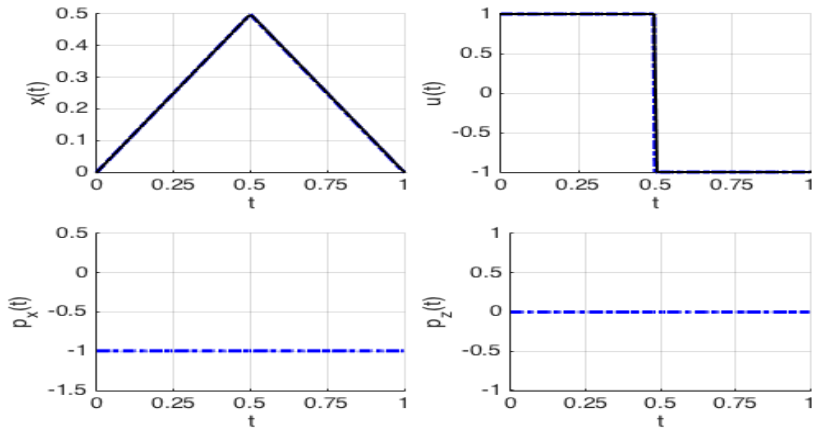


Figure: Case where  $T = 1$ .

# Trajectory optimization problem for a space launcher



## Aim

Maximize the payload  $m_0$  to be steered from the Earth (Kourou) to a prescribed Orbit (SSO, GEO, ...).

Collaborations with Airbus (2013-2018), Cnes (2013-2019), MBDA (2018-2021)

- The physical model involves **6+1** state variables, the position  $\vec{X}$  of the launcher in the 3D space, its velocity  $\vec{V}$  and its mass  $M$ :

$$\mathbf{y} := (\mathbf{X}, \mathbf{V}, \mathbf{M}).$$

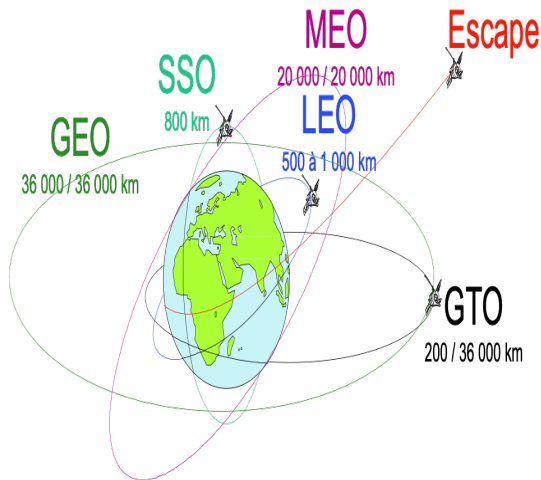
- The forces acting on the rocket are: Gravity  $M\vec{g}$ , Thrust  $\vec{F}_T$ , Drag  $\vec{F}_D$ , and Coriolis forces.
- Newton's Law:

$$\begin{aligned}\frac{d\vec{X}}{dt} &= \vec{V}, \\ \frac{d\vec{V}}{dt} &= \vec{g} + \frac{\vec{F}_T}{M} + \frac{\vec{F}_D}{M} - 2\vec{\Omega} \wedge \vec{V} - \vec{\Omega} \wedge (\vec{\Omega} \wedge \vec{X}),\end{aligned}$$

- The launcher is controlled by means of:
  - *launch parameters*  $\Pi = (\psi, \omega) \in \mathbb{R}^2$
  - *incidence and sideslip angles*  $\alpha(\cdot), \delta(\cdot)$

- Physical state-constraints (intermediate times, end-point)

# Trajectory optimization under final and intermediate constraints (MISTRAL: software developed in coll. with CNES)



# Conclusion

- When the optimal control problem is in presence of state constraints, the value function is merely l.s.c unless some strong controllability assumptions are satisfied.
- The main feature of introducing the auxiliary control problem (with an improvement function) is that its value function is more smooth, it can be easily characterized by a HJB equation.
- The link between the HJB approach and PMP can be established by using the auxiliary function (and not the original discontinuous value function).
- This framework can be extended to multiple intermediate constraints or pointwise state-constraints.

*....thanks for your attention.*