JPR2019

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Alexander von Humboldt Stiftung/Foundation

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JPR1

Feedback boundary stabilization of the two-dimensional Navier--Stokes equations — [PDF] siam.org

JP **Raymond** - SIAM Journal on Control and Optimization, 2006 - SIAM

We study the exponential stabilization of the linearized Navier--Stokes equations around an unstable stationary solution, by means of a feedback boundary control, in dimension 2 or 3. The feedback law is determined by solving a linear-quadratic control problem. We do not …

Hamiltonian Pontryagin's principles for control problems governed by semilinear parabolic equations — [PDF] ensta-paris.fr

JP **Raymond**, H Zidani - Applied Mathematics and Optimization, 1999 - Springer

In this paper we study optimal control problems governed by semilinear parabolic equations. We obtain necessary optimality conditions in the form of an exact Pontryagin's minimum principle for distributed and boundary controls (which can be unbounded) and bounded …

Error estimates for the numerical approximation of Dirichlet boundary control for semilinear elliptic equations — [PDF] siam.org

E Casas, JP **Raymond** - SIAM Journal on Control and Optimization, 2006 - SIAM

We study the numerical approximation of boundary optimal control problems governed by semilinear elliptic partial differential equations with pointwise constraints on the control. The control is the trace of the state on the boundary of the domain, which is assumed to be a …

JPR2

Data driven control

Data-driven control system

From Wikipedia, the free encyclopedia

Data-driven control systems are a broad family of control systems, in which the identification of the process model and/or the design of the controller are based entirely on *experimental data* collected from the plant [1].

In many control applications, trying to write a mathematical model of the plant is considered a hard task, requiring efforts and time to the process and control engineers. This problem is overcome by *data-driven* methods, which allow to fit a system model to the experimental data collected, choosing it in a specific models class. The control engineer can then exploit this model to design a proper controller for the system. However, it is still difficult to find a simple yet reliable model for a physical system, that includes only those dynamics of the system that are of interest for the control specifications. The *direct* data-driven methods allow to tune a controller, belonging to a given class, without the need of an identified model of the system. In this way, one can also simply weight process dynamics of interest inside the control cost function, and exclude those dynamics that are out of interest.

Origins?

Gabriel Peyré
@gabrielpeyre

Oldies but goldies: A. Legendre, Nouvelles méthodes pour la détermination des orbites des comètes, 1805. First publication of the least square method, before Gauss according to French people …
projecteuclid.org/download/pdf_1…

APPENDICE.

Sur la Méthode des moindres quarrés.

Dans la plupart des questions où il s'agit de tirer des mesures données par l'observation, les résultats les plus exacts qu'elles peuvent offrir, on est presque toujours conduit à un système d'équations de la forme

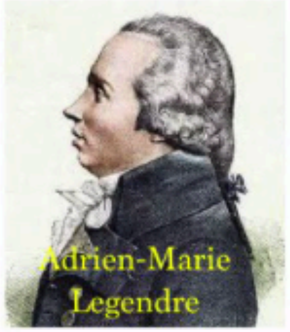
$$E = a + bx + cy + fz + \&c.$$

dans lesquelles a, b, c, f, &c. sont des coëfficiens connus, qui varient d'une équation à l'autre, et x, y, z, &c. sont des inconnues qu'il faut déterminer par la condition que la valeur de E se réduise, pour chaque équation, à une quantité ou nulle ou très-petite.

Si l'on a autant d'équations que d'inconnues x, y, z, &c., il n'y a aucune difficulté pour la détermination de ces inconnues, et on peut rendre les erreurs E absolument nulles. Mais le plus souvent, le nombre des équations est supérieur à celui des inconnues, et il est impossible d'anéantir toutes les erreurs.

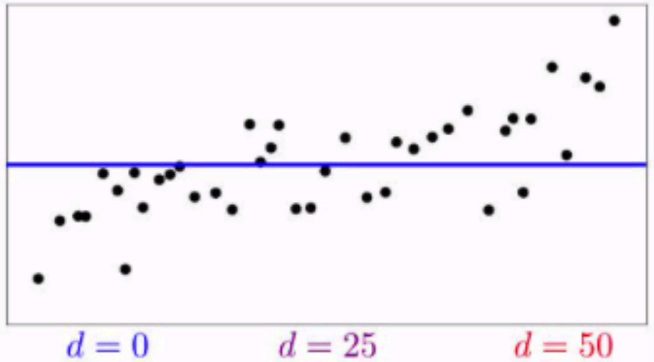
Dans cette circonstance, qui est celle de la plupart des problêmes physiques et astronomiques, où l'on cherche à déterminer quelques élémens importans, il entre nécessairement de l'arbitraire dans la distribution des erreurs, et on ne doit pas s'attendre que toutes les hypothèses conduiront exactement aux mêmes résultats; mais il faut sur-tout faire en sorte que les erreurs extrêmes, sans avoir égard à leurs signes, soient renfermées dans les limites les plus étroites qu'il est possible.

De tous les principes qu'on peut proposer pour cet objet, je pense qu'il n'en est pas de plus général, de plus exact, ni d'une application plus facile que celui dont nous avons fait usage dans les recherches précédentes, et qui consiste à rendre



Least square:

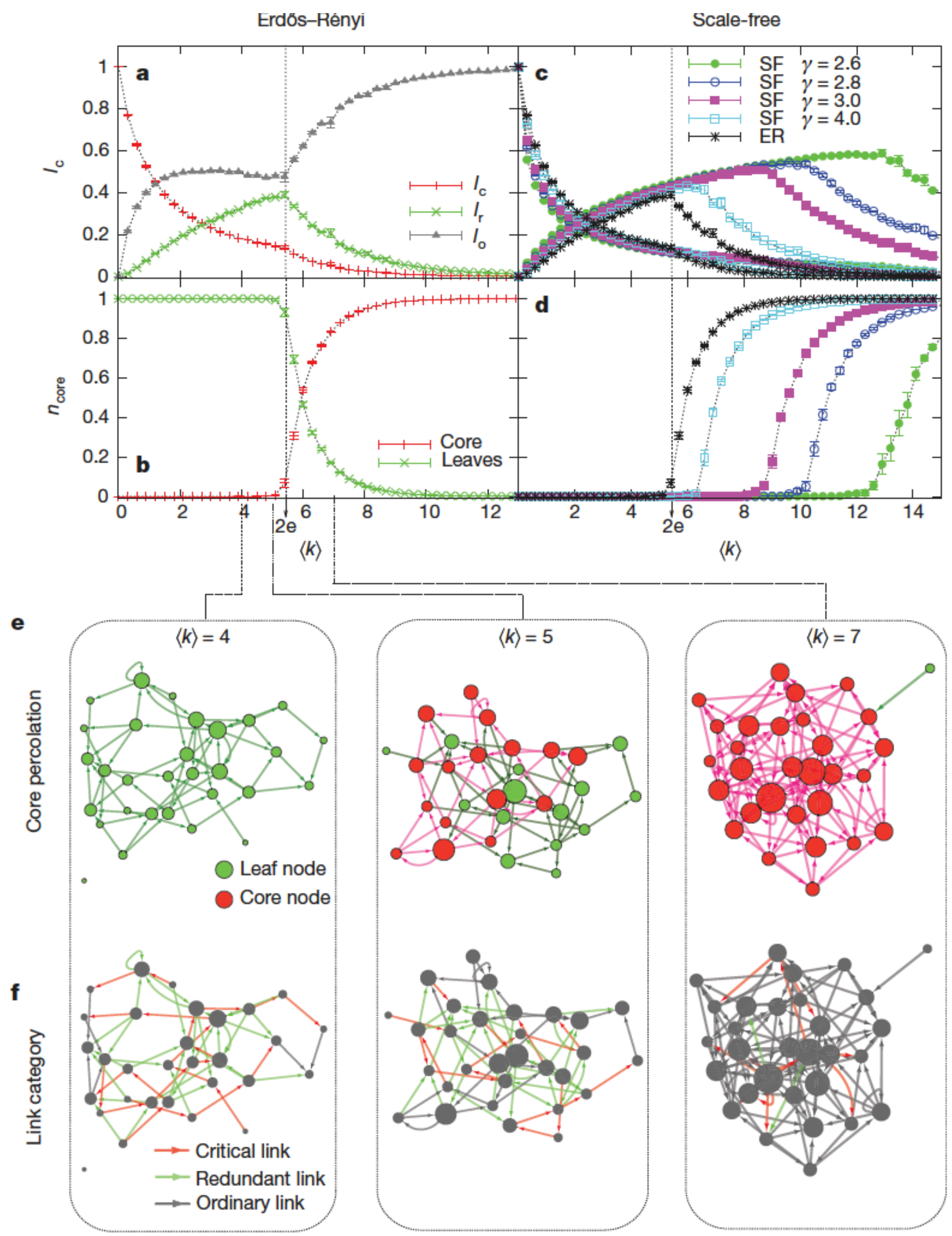
$$\min_{w \in \mathbb{R}^{d+1}} \sum_{i=1}^{n} (y_i - \sum_{k=0}^{d} w_k x_i^k)^2$$



Aug 28, 2019

Complexity

Y.-Y. Liu and A.-L. Barabási, Control principles of complex systems, Rev. Modern Physcis, 2016



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Given a system

$$x' = Ax$$

with $x \in \mathbb{R}^N$, $A \in M(N \times N)$, to find the optimal observer B, in a given class, optimising the observability inequality

$$|x(0)|^2 \leq C(A, B) \int_0^T |Bx|^2 dt.$$

In other words,

$$\min_{B \in \mathcal{C}} C(A, B).$$

In a series of works in collaboration with **Y. Privat and E. Trélat** this problem was addressed in the context of the *heat and wave equation*.

Ingredients:

- Randomise the class of initial data (N. Burq et al.) and consider the reduced/simplified problem:

  $$|e|^2 \leq C_r(A, B)|Be|^2$$

  within the class of eigenfunctions

  $$Ae = \lambda e.$$

- Then address the randomised version of the optimal observer problem

$$\min_{B\in\mathcal{C}} C_r(A,B).$$

- Use the know properties of the eigenfunctions of the Laplacian (in terms of the shape of the domain under consideration).
- Roughly:
  - For heat-like equations optimal observer location is determined by a finite number of low frequency eigenfunctions.
  - For wave-like equations relaxation phenomena may occur.

What about the finite-dimensional problem? How to exploit the structure of the matrix A and the corresponding nature of eigenvectors? How to choose the optimal support of the observation matrix B, when it is simply constituted by 0's and 1's

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Decay for partially dissipative hyperbolic systems

K. Beauchard and E. Z. Sharp large time asymptotics for partially dissipative hyperbolic systems, ARMA, 2011.

$$\frac{\partial w}{\partial t} + \sum_{j=1}^{m} A_j \frac{\partial w}{\partial x_j} = -Bw \; , \; x \in \mathbb{R}^m \; , \; w \in \mathbb{R}^n \qquad (1)$$

$A_1, \ldots, A_m$ symmetric $\;\Big|\;$ $B = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \begin{matrix} \updownarrow n_1 \\ \updownarrow n_2 \end{matrix}$ $\qquad X^t DX > 0 \quad \forall X \in \mathbb{R}^{n_2} - \{0\}$

Goal: Understand the asymptotic behavior as $t \to \infty$.

Apply Fourier transform:

$$\frac{\partial \hat{w}}{\partial t} = (-B - iA(\xi))\hat{w} \qquad \text{where} \qquad A(\xi) := \sum_{j=1}^{m} \xi_j A_j$$

Lack coercivity : $\langle [B + iA(\xi)]X, X\rangle = \langle BX, X\rangle = \langle DX_2, X_2\rangle \not\geqslant c|X|^2$

But possible decay depending on ξ:

$$\exp[(-B - iA(\xi))t] \leqslant Ce^{-\lambda(\xi)t}$$

PARTIALLY DISSIPATIVE LINEAR HYPERBOLIC SYSTEM

$\equiv$

m-PARAMETER (ξ) FAMILY OF FINITE-DIMENSIONAL PARTIALLY DISSIPATIVE n-DIMENSIONAL SYSTEMS.

The asymptotic behavior of solutions is determined by the behavior of the function $\xi \to \lambda(\xi)$ giving the decay rate as a function of ξ.

Roughly, this is related to the dependence on ξ of the best constant $\gamma(\xi)$ such that

$$< Be, e > \geq \gamma(\xi)|e|^2$$

for all eigenvectors

$$A(\xi)e = \mu(\xi)e.$$

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Collective behavior models

- Describe the dynamics of a system of interacting individuals.
- Applied in a large spectrum of subjects such as collective behavior, synchronization of coupled oscillators, random networks, multi-area power grid, opinion propagation,...

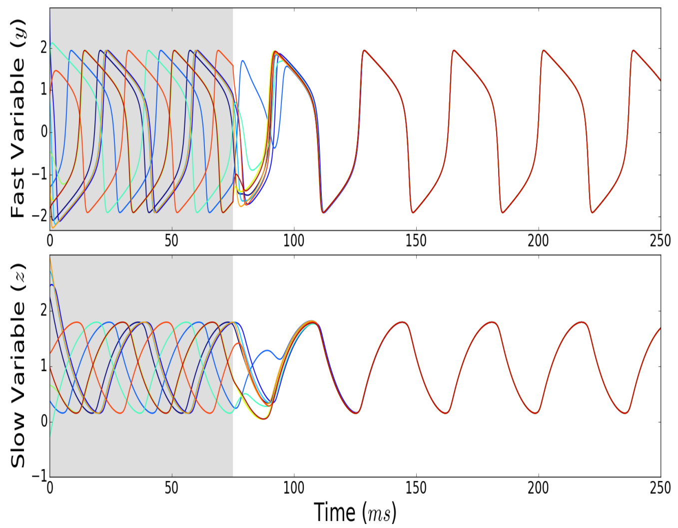

Figure: Fitz-Hugh-Nagumo oscillators [Davison et al., Allerton 2016]

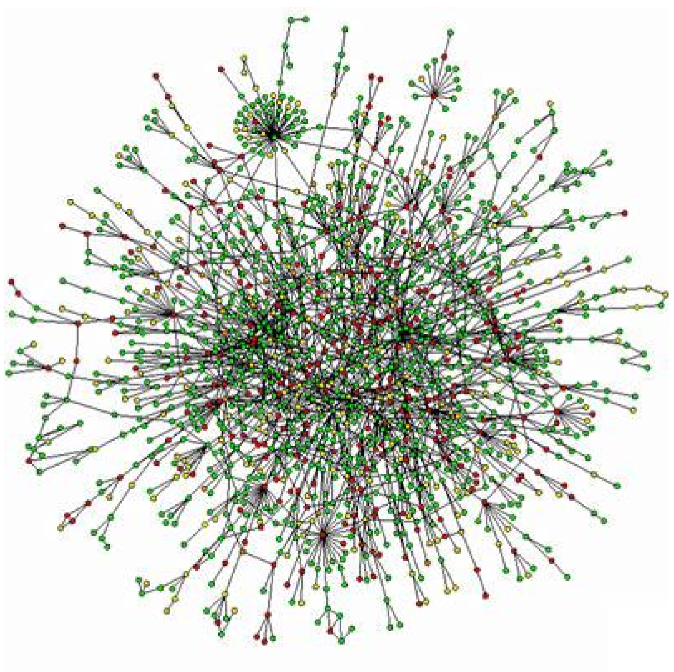

Figure: Yeast's protein interactions [Jeong et al., Nature, 2001]

Figure: French electric network

Some basic references on the Dynamics and Control on networks and graphs

[1] Kuramoto, Y. (1984). Chemical Oscillations, Waves, and Turbulence. Springer-Verlag Berlin Heidelberg.

[2] Olfati-Saber, R., Fax, J. A. & Murray, R. M. Consensus and cooperation in networked multi-agent systems. IEEE Proc. 95, 1 (2007), 215–233.

[2] Y.-Y Liu, J.-J. Slotine & A.-L. Barabási, Controllability of Complex Networks, Nature, 473, 167–173 (12 May 2011).

[3] T. Vicsek & A. Zafeiris, Collective motion, Physics Reports 517 (2012) 71–140.

[4] S. Motsch & E. Tadmor. Heterophilious dynamics enhances consensus. SIAM Review 56, 4 (2014), 577–621.

...

And many others[1] [2]

[1] M. Caponigro, M. Fornasier, B Piccoli & E. Trélat, M3AS, 2015
[2] M. Burger, R. Pinnau, A. Roth, C. Totzeck & O. Tse, arXiv 2016.

Complex behavior by simple interaction rules

Systems of Ordinary Differential Equations (ODEs) in which each agent's dynamics follows a prescribed law of interactions:

First-order consensus model

$$\dot{x}_i(t) = \frac{1}{N}\sum_{j=1}^{N} a_{i,j}(x_j(t) - x_i(t)), \quad i = 1, \dots, N$$

- It describes the opinion formation in a group of N individuals.
- $x_i \in \mathbb{R}^d$, $d \geq 1$, represents the opinion of the i-th agent.

  [J. R. P. French, A formal theory of social power, Psychol. Rev., 1956].
- It applies in several fields including information spreading of social networks, distributed decision-making systems or synchronizing sensor networks, ...

From random to consensus

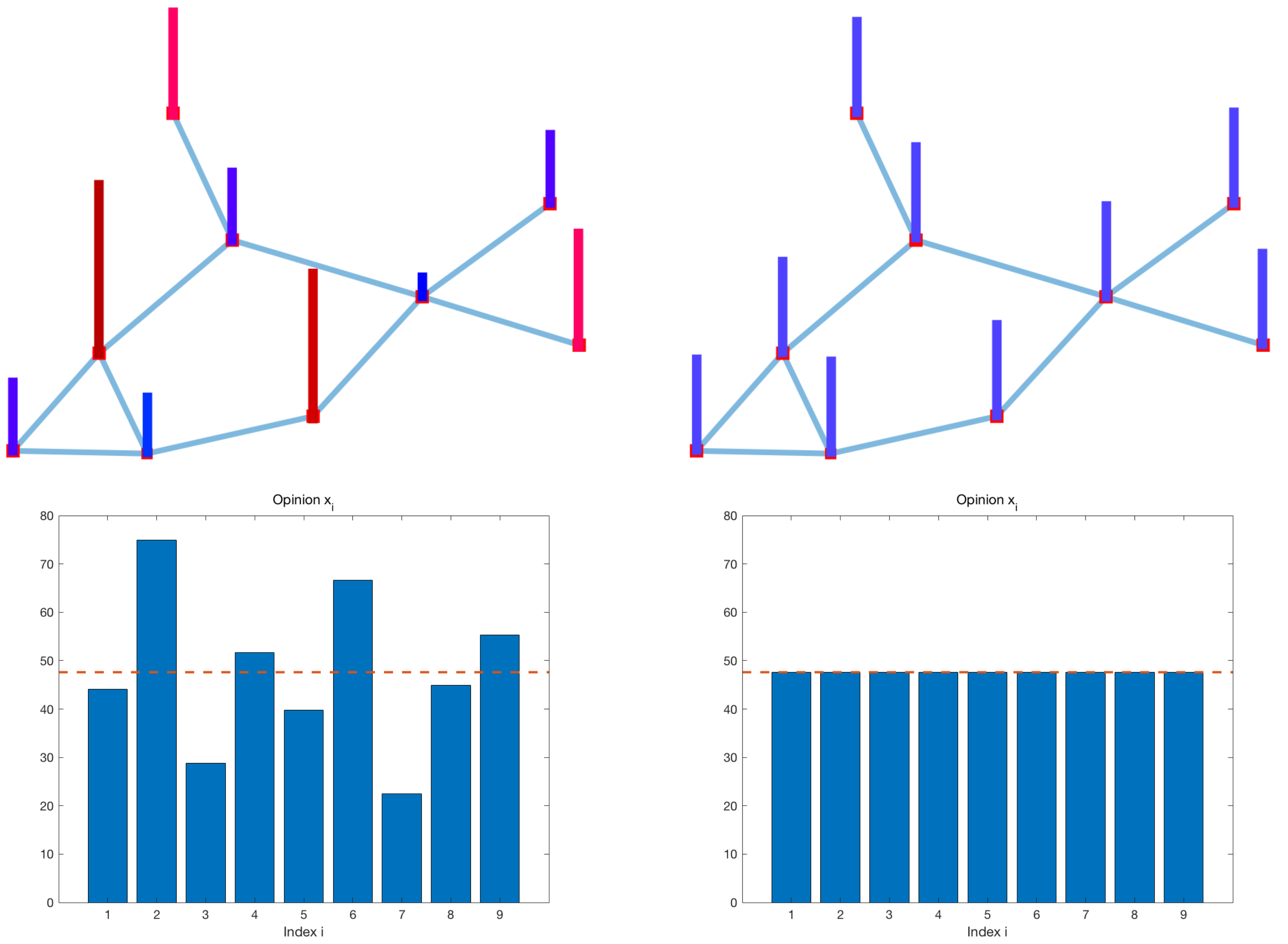

Figure: Opinions over a network : random versus consensus states

Linear versus Nonlinear

- Linear networked multi-agent models: $a_{i,j}$ are the elements of the adjacency matrix of a graph with nodes x_i

$$a_{i,j} := \begin{cases} a_{j,i} > 0, & \text{if } i \neq j \text{ and } x_i \text{ is connected to } x_j \\ 0, & \text{otherwise.} \end{cases}$$

  This leads to the semi-discrete heat equation on the graph.

- Nonlinear alignment models:

$$a_{i,j} := a(|x_j - x_i|), \quad \text{where} \quad a : \mathbb{R}_+ \to \mathbb{R}_+,$$

  $a \geq 0$ is the influence function. The connectivity depends on the contrast of opinions between individuals.

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Limitation of the mean-field representation

As the number of agents $N \to \infty$, ODE $\to$ PDE.

- Nonlinear alignment models:

$$\dot{x}_i = \frac{1}{N}\sum_{j=1}^{N} a(|x_j - x_i|)(x_j - x_i), \quad i = 1, \dots, N, \quad a : \mathbb{R}_+ \to \mathbb{R}_+.$$

Classical **mean-field theory**: Define the N-particle distribution function[3]

$$\mu^N = \mu^N(x,t) := \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i(t)}.$$

and let $N \to +\infty$.

[3] P. A. Raviart, Particle approximation of first order systems, J. Comp. Math., 4 (1) (1986), 50-61.
By particle methods of approximation of time-dependent problems in PDE, we mean numerical methods where, for each time t, the exact solution is approximated by a linear combination of Dirac measures...

- The limit μ of the empirical measures μ^N solves the the nonlocal transport equation[4]

$$\partial_t \mu(x,t) = \partial_x \Big(\mu(x,t) V[\mu(x,t)] \Big)$$

$$V[\mu](x,t) := \int_{\mathbb{R}^d} a(|x-y|)(x-y)\mu(y,t)dy.$$

The convolution kernel describes the mixing of opinions by the interaction of agents along time.

- In other words:[5]

$$\partial_t \mu = \partial_x \left(\mu(x,t) \int_{\mathbb{R}^d} a(|x-y|)(x-y)\mu(y,t)\, dy \right).$$

[4] The system of ODEs describing the agents dynamics defines the characteristics of the underlying transport equation. The coupling of the agents dynamics introduces the non-local effects on trasport.

[5] Motsch and Tadmor, SIAM Rev., 2014

The mean field model does not track individuals!

- The mean-field equation involves the density μ, which does not contain the full information of the state.
- The density μ does not keep track of the identities of agents (label i).[6] Different configurations x_i with the same distribution μ

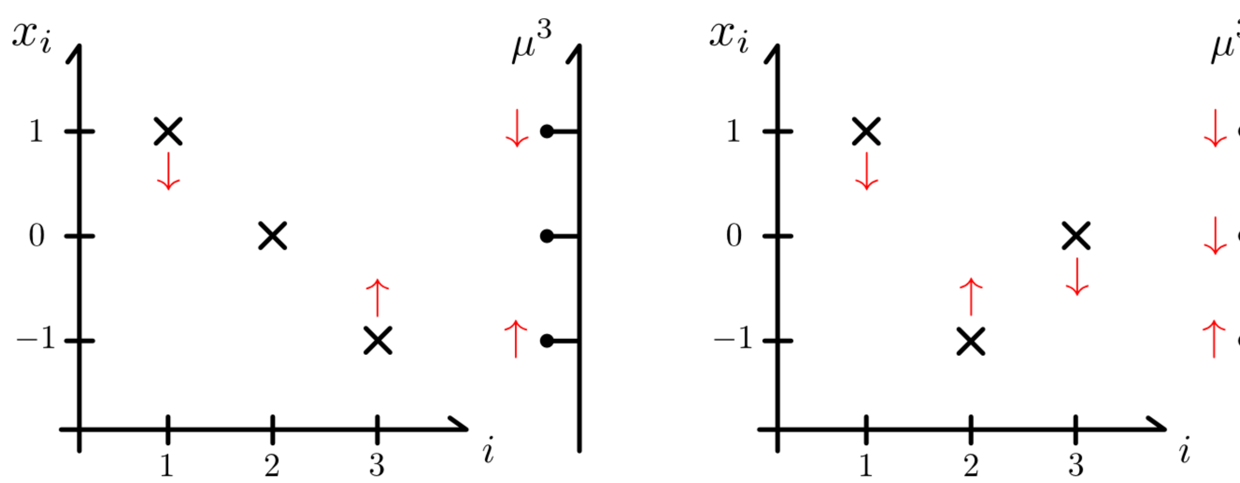

Figure: $x^1 = (-1, 0, 1)$ (left) and $x^2 = (-2, 3, -1)$ (right) generate the same density function.

[6] $\mu^N(x) := \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i}$

Graph limit method: finite-difference approach

- Based on the theory of graph limits (Medvedev, SIAM J. Math. Anal., 2014).
- Considering the phase-value function $x^N(s,t)$ defined as

$$x^N(s,t) = \sum_{i=1}^{N} x_i(t)\chi_{I_i}(s,t), \quad s \in (0,1),\ t > 0,\ \bigcup_{i=1}^{N} I_i = [0,1].$$

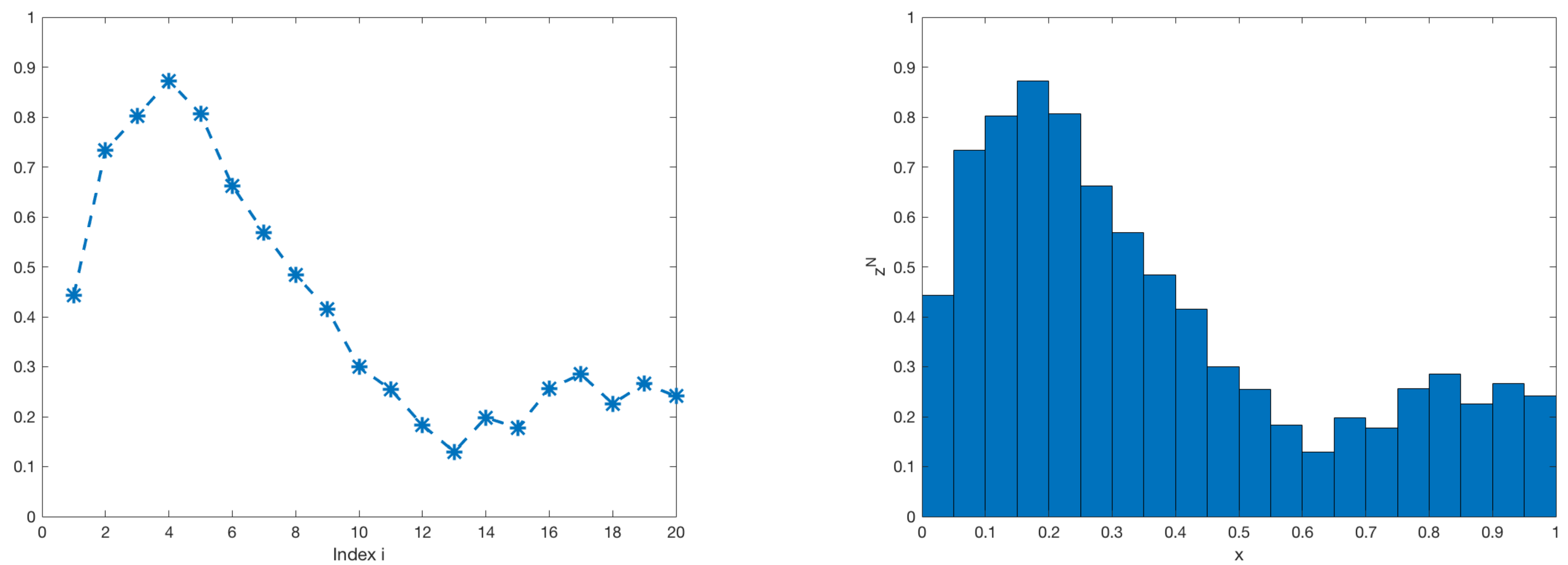

Figure: Opinion ($N = 20$) and its finite-difference function z^{20} on $[0,1]$

- Let $(x_i^N)_{i=1}^N$ be the solution of the following consensus model:

$$\dot{x}_i^N = \frac{1}{N}\sum_{j=1}^{N} a_{i,j}^N \psi(x_j^N - x_i^N),$$

where $a_{i,j}^N$ are constant and ψ represents nonlinearity.

- According to the graph limit theory[7], if

$$W^N(s,s_*) = \sum_{i,j=1}^{N} a_{i,j}^N \mathbf{1}_{[\frac{i}{N},\frac{(i+1)}{N})}(s)\mathbf{1}_{[\frac{j}{N},\frac{(j+1)}{N})}(s_*)$$

is uniformly bounded and converges to W, then in the limit $N \to \infty$ we get the non-local diffusive equation,

$$\partial_t x(s,t) = \int_{[0,1]} W(s,s_*)\psi(x(s_*,t) - x(s,t))ds_*.$$

[7] G. S. Medvedev. SIAM J. Math. Anal. 46, 4 (2014), 2743–2766.

Nonlinear subordination

U. Biccari, D. Ko & E. Z., M3AS, 2019.

- $$\dot{x}_i = \frac{1}{N}\sum_{j=1}^{N} a(|x_j - x_i|)(x_j - x_i).$$
- The **Graph limit** model:
$$x_t(s,t) = \int_{[0,1]} a(|x(s_*,t) - x(s,t)|)(x(s_*,t) - x(s,t))ds_*.$$
- The **mean-field limit**:
$$\mu_t(x,t) + \nabla_x(V[\mu]\mu) = 0, \quad \text{where} \quad V[\mu] := \int_X a(x_* - x)\mu(x_*,t)dx_*.$$

Subordination transformation

From non-local "parabolic" to non-local "hyperbolic":
$\mu(x,t) = \int_S \delta(x - x(s,t))ds.$

8

[8]Kinetic / conservation laws: Lions-Perthame-Tadmor, JAMS, 1994

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Control (N fixed)

- **Linear**: Classical Kalman rank conditions.
- **Nonlinear**: controllability and stabilisability is a much more subtle issue [9] is much more challenging

$$\begin{cases} \dot{x}(t) = \frac{1}{N}\sum_{j=1}^{N} a_{i,j}(x_j(t) - x_i(t)) + \sum_{j=1}^{M} b_{i,j}u_j(t), \quad i = 1,\dots,N, \\ x(0) = x_0, \end{cases}$$

- The linear model can be viewed as the **linearisation** of the nonlinear one on the consensus configuration.

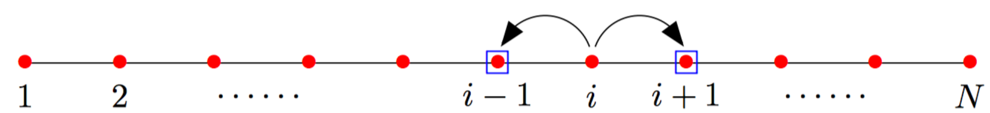
Many different issues arise:

- by acting on all the components of the system (certainly effective but not always optimal).
- by focusing only on a small number of agents at each time (sparse control).
- by looking for a single leader who acts on the whole crowd and steers it to the desired configuration (control through leadership).

[9] Caponigro, Fornasier, Piccoli, and Trélat, Math. Models Methods Appl. Sci., 2015

Linear networks as finite-dim. approximations of PDEs

$$a_{i,j} = \begin{cases} 1, & \textit{if } j = i \pm 1 \\ 0, & \textit{otherwise} \end{cases}$$



$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \vdots \\ \vdots \\ \dot{x}_N \end{pmatrix} + \color{red}{\frac{1}{N}} \underbrace{\begin{pmatrix} 1 & -1 & 0 & \dots & \dots & 0 \\ -1 & 2 & -1 & \dots & \dots & 0 \\ \vdots & & \ddots & & & \vdots \\ \vdots & & & \ddots & & \vdots \\ 0 & \dots & \dots & -1 & 2 & -1 \\ 0 & \dots & \dots & \dots & -1 & 1 \end{pmatrix}}_{L} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{pmatrix}.$$

Rescaled version of the finite-difference approximations of the heat equation!

The system is a rescaled version of the finite difference semi-discretization of the one-dimensional heat equation with homogeneous Neumann boundary conditions on $[0,1]$.

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \vdots \\ \vdots \\ \dot{x}_N \end{pmatrix} + N^2 \underbrace{\begin{pmatrix} 1 & -1 & 0 & \dots & \dots & 0 \\ -1 & 2 & -1 & \dots & \dots & 0 \\ \vdots & & \ddots & & & \vdots \\ \vdots & & & \ddots & & \vdots \\ 0 & \dots & \dots & -1 & 2 & -1 \\ 0 & \dots & \dots & \dots & -1 & 1 \end{pmatrix}}_{D} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{pmatrix}.$$

The only difference is in the scale with respect to N.
The network system corresponds to the finite-difference discretisation of

$$u_t - N^{-3} u_{xx} = 0.$$

Controllability cost

Acting locally in one individual to achieve a global goal

- The cost of controlling the system depends in T and N so that
$$\mathcal{C} \sim \exp(N^3/T).$$

- One may keep it bounded by setting
$$T \sim N^3.$$

- Developing a complete theory of controllability for linear discrete models as $N \to \infty$ is a very challenging topic.
- The existing theory for the null-control of semi-discrete versions of the heat equation is rather limited, based on **spectral analysis** (very particular cases) or in **Carleman inequalities** and/or multiplier identities for wave equations plus transmutation arguments. And the later ones led to high frequency numerical reminder terms. [10] [11] [12]

[10] Nonlinear versions developed in collaboration with Domenec Ruiz

[11] E. Z., Proceedings of the ICM Madrid ,2006.

[12] F. Boyer &J. LeRousseau, Ann. I.H. Poincaré – AN, 31(2014)1035 – 1078.

The fractional heat equation

The same analysis can be carried out for other discrete systems, leading, in particular, to fractional heat equations

$$u_t + (-\partial_x^2)^\alpha u = 0, \quad t \geq 0,$$

$$(-\partial_x^2)^\alpha = c(\alpha) P.V. \int_{-\infty}^{+\infty} \frac{(u(x) - u(y))}{|x-y|^{1+2\alpha}} dy$$

This is an example of the all-to-all networked models. The spectrum

$$\lambda_k^{frac,\alpha} \sim k^{2\alpha}, \quad k \geq 1.$$

Null-controllability conditions hold if and only if $\alpha > 1/2$.

This corresponds to the following weighted all-to-all networked model: [13] [14]

$$a_{i,j} = \frac{c(\alpha)}{[|i-j|/N|]^{1+2\alpha}}, if\ j \neq i.$$

[13] S. Micu & E. Z. SIAM J. Cont. Optim., 44(6) (2006) 1950-1972.
[14] U. Biccari, V. Hernández-Santamaria. Controllability of a one-dimensional fractional heat equation: theoretical and numerical aspects. 2017. <hal-01562358v2>

Merci Jean-Pierre, pour ton amitit ta collaboration loyale. Et trbonne continuation!

JPR on passive mode

CIMI Chair 2014

Alexander von Humboldt Stiftung/Foundation erc