Sparse optimal control for a nonlinear parabolic equation with mixed control-state constraints

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Control and stabilization issues for PDE

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Chemnitz-Klaffenbach 2011

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Outline

Sparse optimal control problem

$$\min J(y,u) := \frac{1}{2}\int_Q |y-y_Q|^2\,dxdt + \frac{\nu}{2}\int_\Sigma |u|^2\,d\sigma dt + \kappa\int_\Sigma |u|\,d\sigma dt$$

subject to

$$\begin{aligned} \partial_t y - \Delta y + d(y) &= 0 && \text{in } Q := \Omega\times(0,T)\\ \partial_n y + b(y) &= u && \text{in } \Sigma := \Gamma\times(0,T)\\ y(x,0) &= y_0 && \text{in } \Omega, \end{aligned}$$

$$u_a \le u(x,t) \le u_d + y(x,t)$$

for a.a. $(x,t)\in\Sigma$.

Data

$y_Q \in L^\infty(Q)$, $T>0$, $\Omega\subset\mathbb{R}^N$, $N\le 3$, bounded Lipschitz domain; ∂_n :outward normal derivative; $\nu>0$, $\kappa>0$, $u_a < 0 < u_d$, assume $y_0 = 0$ for the talk.

State: $y \in W(0,T)\cap L^\infty(Q)$

Control: $u\in L^\infty(\Sigma)$.

Main difficulties

$$\min J(y,u) := \frac{1}{2}\int_Q |y - y_Q|^2 dx\,dt + \frac{\nu}{2}\int_\Sigma |u|^2\, d\sigma dt + \kappa \int_\Sigma |u|\, d\sigma dt$$

subject to

$$\begin{aligned} \partial_t y - \Delta y + d(y) &= 0 \quad \text{in } Q := \Omega \times (0,T) \\ \partial_n y + b(y) &= u \quad \text{in } \Sigma := \Gamma \times (0,T) \\ y(x,0) &= 0 \quad \text{in } \Omega, \end{aligned}$$

$$u_a \le u(x,t) \le u_d + y(x,t).$$

- Nondifferentiability of $\|u\|_{L^1(\Sigma)}$
- Nonlinearity of the control-to-state mapping $u \mapsto y$ (nonconvex problem)
- Existence of a Lagrange multiplier μ to $u - y - u_d \le 0$

Outline

Well-posedness of the state equation

Assumption (Simplified)

- *d and b are real functions of class C^1 with locally Lipschitz derivative*
- *There is $c_0 \in \mathbb{R}$: $d'(y) \geq c_0$ and $b'(y) \geq c_0 \quad \forall y \in \mathbb{R}$.*

Example: $d(y) = (y+1)(y-1)(y-2), \qquad b(y) = y^3|y|$

(FHN equations, Stefan-Boltzmann radiation condition)

Theorem

For every $u \in L^p(\Sigma)$ with $p > N+1$, the state equation has a unique solution $y_u \in W(0,T) \cap L^\infty(Q)$.

We denote by y_u the state associated with u.

Inverse isotony

Notation: For u, $v \in L^2(\Sigma)$, we write $u \leq v$ if $u(x,t) \leq v(x,t)$ holds a.e. in Σ.

Theorem (E. Casas, F.T. 2018)

For every function $g \in L^p(\Sigma)$ with $p > N+1$, the equation

$$v - y_v = g \quad \text{in } L^p(\Sigma)$$

has a unique solution $v \in L^p(\Sigma)$.
If $u \in L^p(\Sigma)$ satisfies the inequality

$$u - y_u \leq g \quad \text{in } L^p(\Sigma),$$

then $u \leq v$ holds.

Control-to-state mapping: $G : u \mapsto y_u$, $G(u) = y_u$, $\quad G_\Sigma : u \mapsto y_u|_\Sigma$.

The theorem above in a nutshell:

$$u - G_\Sigma(u) \leq v - G_\Sigma(v) \quad \Longrightarrow \quad u \leq v \qquad \text{(inverse isotony)}$$

Existence of an optimal control

Lemma

There is some constant $M > 0$ such that

$$u_a \le u \le M$$

holds for all feasible controls of the optimal control problem.

The proof is based on inverse isotony:

$$u \le u_d + y_u \quad \Leftrightarrow \quad u - G_\Sigma(u) \le u_d$$

$$\Rightarrow \quad u \le z, \text{ where } z \in L^\infty(\Sigma) \text{ solves } \; z - G_\Sigma(z) = u_d; \; \text{take } M := \|z\|_{L^\infty(\Sigma)}.$$

Theorem

Assume that there exists a feasible control, i.e. a control with $u_a \le u \le u_d + y_u$. Then there exists at least one optimal control $\bar{u}$.

Differentiability

Theorem

The mapping G is of class C^1. The derivative of G at $u \in L^p(\Sigma)$ in the direction $v \in L^p(\Sigma)$ is given by

$$G'(u)v = z_v,$$

where $z_v \in W(0,T) \cap L^\infty(Q)$ is the unique solution to the linearized equation

$$\begin{aligned} \partial_t z - \Delta z + d'(y_u)\, z &= 0 \\ \partial_n z + b'(y_u)\, z &= v \\ z(x,0) &= 0. \end{aligned}$$

By inverse isotony, we show that $G'(u)$ is a nonnegative operator, i.e.

$$v \geq 0 \;\Rightarrow\; G'(u)v \geq 0.$$

Outline

Expected optimality conditions

$$\min J(u) := \frac{1}{2}\int_Q |\underbrace{y_u}_{G(u)} - y_Q|^2\,dxdt + \frac{\nu}{2}\int_\Sigma |u|^2\,d\sigma dt + \kappa\int_\Sigma |u|\,d\sigma dt$$

subject to

$$\begin{aligned} \partial_t y - \Delta y + d(y) &= 0 \\ \partial_n y + b(y) &= u \\ y(x,0) &= 0, \end{aligned}$$

$$u_a \le u \le u_d + y_u.$$

Definition:

$$f(u) = \frac{1}{2}\int_Q |G(u) - y_Q|^2\,dxdt + \frac{\nu}{2}\int_\Sigma |u(x,t)|^2\,d\sigma dt,$$

$$j(u) = \|u\|_{L^1(\Sigma)}.$$

Reduced optimal control problem

The optimal control problem is equivalent to

$$\min_{u\in F} J(u) := f(u) + \kappa j(u) \qquad \text{(P)}$$

with the *feasible (in general non-convex) set* $F \subset L^\infty(\Sigma)$ defined by

$$F := \{u \in L^\infty(\Sigma) : \quad u_a \le u \le u_d + G_\Sigma(u)\}.$$

Lagrangian function

$$\mathcal{L}(u,\mu) := f(u) + \kappa j(u) + \int_\Sigma (u - G_\Sigma(u) - u_d)\,\mu\, d\sigma dt.$$

If a Lagrange multiplier $0 \le \mu \in L^2(\Sigma)$ exists then an optimal solution $\bar u$ of (P) should satisfy the optimality conditions of the problem without state constraint

$$\min_{u\ge u_a} \mathcal{L}(u,\mu).$$

"Sum of a nonconvex differentiable and a convex nondifferentiable functional"

Roadmap for optimality conditions

- Linearization of the equation $\longrightarrow$ Convex control problem with linear mixed control-state-constraint $\leftarrow$ Linearization theorem
- Subdifferential calculus $\longrightarrow$ Linearized control problem with linear mixed control-state-constraint
- Transfer of the problem to an L^2-setting
- Duality theory of linear programing to prove the existence of a Lagrange multiplier μ
- Proof of a pointwise minimum principle
- Sparsity theorem (optimal control vanishes in some parts of Σ)

Linearization theorem

Abstract optimization problem: $\min\{f(u) + g(u) : u \in K \text{ and } H(u) \in C\}.$

- U, Y: normed vector spaces,
- $K \subset U$, $C \subset Y$: convex sets
- $H : U \to Y$, $f : U \to \mathbb{R}$, and $g : U \longrightarrow (-\infty, +\infty]$: given mappings.

Theorem (E. Casas, F.T. 2018)

Let $\bar{u}$ be an associated local solution. Assume that f and H are Fréchet differentiable at $\bar{u}$, g is convex, and $\text{int}\, C \neq \emptyset$. If the linearized Slater condition

$$\exists u_0 \in K : H(\bar{u}) + H'(\bar{u})(u_0 - \bar{u}) \in \text{int}\, C$$

is satisfied, then

$$f'(\bar{u})(u - \bar{u}) + g(u) - g(\bar{u}) \geq 0$$

holds for all $u \in K$ that satisfy

$$H(\bar{u}) + H'(\bar{u})(u - \bar{u}) \in C.$$

Linearized optimal control problem

We apply the theorem in $U = Y = L^\infty(\Sigma)$; notice: int $L^\infty(\Sigma)_+ \neq \emptyset$.

Assumption (Linearized Slater condition)

Let $\bar u \in L^\infty(\Sigma)$ be fixed. Assume the existence of $u_0 \in L^\infty(\Sigma)$ and $\delta > 0$ such that

$$u_0 \geq u_a \quad \text{and} \quad u_0 - u_d - y_{\bar u} - G'_\Sigma(\bar u)(u_0 - \bar u) \leq -\delta.$$

From the abstract linearization theorem, we obtain:

Theorem (Linearization of the problem, E.Casas, F.T. 2018)

Let a locally optimal $\bar u \in L^\infty(\Sigma)$ satisfy the linearized Slater condition. Then

$$f'(\bar u)(u - \bar u) + \kappa\, j(u) - \kappa\, j(\bar u) \geq 0$$

for all $u \in L^\infty(\Sigma)$ with

$$u_a \leq u \leq u_d + y_{\bar u} + G'_\Sigma(\bar u)(u - \bar u).$$

Advantage: This is a convex problem, where subdifferential calculus applies.
Drawback: It is yet posed in $L^\infty(\Sigma)$ and $\bar\mu \in L^\infty(\Sigma)^*$ is not a good choice.

Roadmap for completing the necessary conditions

- We consider the linearized control problem in $L^2(\Sigma)$ instead of $L^\infty(\Sigma)$.
- Also in the linearized problem, we have uniform boundedness of all feasible controls. The feasible set does not change by extending it to L^2!
- However, this brings a new difficulty: The interior of the cone of nonnegative functions of $L^2(\Sigma)$ is empty.
- We find a Lagrange multiplier $\bar{\mu} \in L^2(\Sigma)$ as solution of a dual linear programming problem posed in $L^2(\Sigma)$.
- We can even show $\bar{\mu} \in L^\infty(\Sigma)$ and uniform boundedness of $\bar{\mu}$ w.r. to κ.

Definition

The operator $G'_\Sigma(\bar{u}) : L^\infty(\Sigma) \to L^\infty(\Sigma)$ can be continuously extended to $L^2(\Sigma)$. We denote this extension by S; in this extended sense we have

$$S\,u := G'_\Sigma(\bar{u})\,u \quad \forall u \in L^2(\Sigma).$$

Necessary conditions with subdifferential calculus

The control $\bar{u}$ solves $\min_{u_a \le u_d + y_{\bar{u}} + S(u-\bar{u})} \{f'(\bar{u})(u-\bar{u}) + g(u)\}$

Lemma (Use of subdifferential calculus)

Under the linearized Slater condition, there is some $\bar{\lambda} \in \partial j(\bar{u})$ such that

$$f'(\bar{u})(u-\bar{u}) + \kappa \int_\Sigma \bar{\lambda}(x,t)(u(x,t) - \bar{u}(x,t))\, d\sigma dt \ge 0$$

is satisfied for all controls u of $L^2(\Sigma)$ that obey the linearized restrictions

$$u \ge u_a \quad \text{and} \quad u \le u_d + y_{\bar{u}} + S\,(u - \bar{u}).$$

Notice that this is a linear problem!

We recall that $\bar{\lambda}$ belongs to the subdifferential $\partial j(\bar{u})$ iff

$$\bar{\lambda}(x,t) = \begin{cases} 1 & \text{if} \quad \bar{u}(x,t) > 0 \\ \in [-1,1] & \text{if} \quad \bar{u}(x,t) = 0 \\ -1 & \text{if} \quad \bar{u}(x,t) < 0. \end{cases}$$

Next step: Express $f'(\bar{u})(u-\bar{u})$ by an adjoint state.

Adjoint state

Adjoint equation

$$
\begin{aligned}
-\partial_t\bar\varphi - \Delta\bar\varphi + d'(\bar y)\,\bar\varphi &= \bar y - y_Q && \text{in } Q\\
\partial_n\bar\varphi + b'(\bar y)\bar\varphi &= 0 && \text{in } \Sigma\\
\bar\varphi(x,T) &= 0 && \text{in } f\Omega.
\end{aligned}
$$

With this adjoint state $\bar\varphi$, we have

$$
f'(\bar u)(u-\bar u) = \int_\Sigma (\bar\varphi + \nu\bar u)(u-\bar u)\,d\sigma dt.
$$

Preliminary optimality conditions

Theorem (Preliminary necessary condition)

Let $\bar{u}$ be a local solution to the optimal control problem that satisfies the linearized Slater condition. Then there are $\bar{\lambda} \in \partial j(\bar{u})$ and a unique adjoint state $\bar{\varphi} \in L^\infty(\Sigma)$ such that

$$\int_\Sigma (\bar{\varphi}(x,t) + \nu \bar{u}(x,t) + \kappa \bar{\lambda}(x,t))(u(x,t) - \bar{u}(x,t))\, d\sigma dt \geq 0$$

holds for all $u \in L^2(\Sigma)$ that satisfy

$$u_a \leq u \leq u_d + y_{\bar{u}} + S\,(u - \bar{u}).$$

Why only preliminary?

The upper constraint for u is not a pointwise one, it is still an ”operator inequality”.

Existence of a Lagrange multiplier

Assumption (Feasibility of u_a)

The lower bound u_a is assumed to be feasible for the linearized problem, i.e. there holds

$$u_a \le u_d + y_{\bar u} + S(u_a - \bar u).$$

Theorem (Proof by duality theory)

Let the assumption above be fulfilled and let $\bar u \in L^2(\Sigma)$ solve the linearized control problem. Denote by $\bar y := y_{\bar u}$ the associated state. Then there exists a non-negative Lagrange multiplier $\bar\mu \in L^\infty(\Sigma)$ associated with the mixed control-state constraint. This is a function $\bar\mu$ such that

$$\int_\Sigma (\bar\varphi + \nu\,\bar u + \kappa\,\bar\lambda + \bar\mu - S^*\bar\mu)(u - \bar u)\,d\sigma dt \ge 0 \quad \forall u \ge u_a$$

and the following complementarity conditions are satisfied

$$\bar\mu \ge 0, \quad \bar u - \bar y - u_d \le 0, \quad \int_\Sigma (\bar u - \bar y - u_d)\,\bar\mu\,d\sigma dt = 0.$$

Pointwise minimum principle

Yet, the optimality condition cannot be directly applied to proving sparsity properties. We slightly reformulate them to get a pointwise version:

Theorem (Pointwise minimum principle)

Let the last assumption be satisfied. If $\bar{u}$ is a locally optimal control and $\bar{\mu} \in L^\infty(\Sigma)$ is an associated Lagrange multiplier that exists according to the last theorem, then for almost all $(x, t) \in \Sigma$ the solution u of the problem

$$\min \, (\underbrace{\bar{\varphi} - S^* \bar{\mu}}_{\bar{\psi}} + \nu \, \bar{u} + \kappa \, \bar{\lambda})(x, t) \, u$$

subject to $u \in \mathbb{R}$ and

$$u_a \leq u \leq u_d + \bar{y}(x, t)$$

is attained by the real number $u = \bar{u}(x, t)$.

Extended adjoint state

We have $\bar\varphi|_\Sigma - S^*\bar\mu = \bar\psi|_\Sigma$ where the extended adjoint state $\bar\psi$ solves

$$\begin{aligned} -\partial_t\psi - \Delta\psi + d'(\bar y)\,\psi &= \bar y - y_Q \\ \partial_n\psi + b'(\bar y)\,\psi &= -\bar\mu \\ \psi(x,T) &= 0. \end{aligned}$$

Pointwise minimum principle

$$\min_{u_a \le u \le u_d + \bar y(x,t)} (\bar\psi(x,t) + \nu\,\bar u(x,t) + \kappa\,\bar\lambda(x,t))\,u$$

is attained by $u = \bar u(x,t)$ a.e. on Σ.

For sparsity, we need $u_a < 0$ and $u_d + \bar y(x,t) > 0$ ...

Outline

Lemma

Let $u = 0$ be feasible, denote by $\bar{u}_\kappa$ a family of optimal controls corresponding to $\kappa \geq 0$, and let $\bar{y}_\kappa = G(\bar{u}_\kappa)$ be the associated states. Then

$$\lim_{\kappa \to \infty} \|\bar{u}_\kappa\|_{L^p(\Sigma)} = 0 \quad \forall p \in [1, \infty)$$

and hence $\bar{y}_\kappa \to y^0$ in $L^2(0, T; H^1(\Omega)) \cap L^\infty(Q)$, where $y^0 = G(0)$.

Main idea: Assume for simplicity that d and b are monotone; $u = 0$ is feasible for all κ, and then $y = 0$ is the associated state. Therefore,

$$\begin{aligned} f(\bar{u}_\kappa) + \kappa\, j(\bar{u}_\kappa) &= \frac{1}{2}\|\bar{y}_\kappa - y_Q\|^2_{L^2(Q)} + \frac{\nu}{2}\|\bar{u}_\kappa\|^2_{L^2(\Sigma)} + \kappa\,\|\bar{u}_\kappa\|_{L^1(\Sigma)} \\ &\leq f(0) + \kappa j(0) = \frac{1}{2}\|y_Q\|^2_{L^2(Q)}. \end{aligned}$$

This immediately yields $\|\bar{u}_\kappa\|_{L^1(\Sigma)} \to 0$. Uniform boundedness $\Rightarrow$ $\|\bar{u}_\kappa\|_{L^p(\Sigma)} \to 0$, hence $(\bar{y}_\kappa)_{|\Sigma} \to 0$ in $L^\infty(\Sigma)$. $\square$

This shows that $(u_d + \bar{y}_\kappa)(x, t) \geq \delta > 0 \quad \forall \kappa \geq \kappa_0,$ a.e. in Σ.

Sparsity

Theorem (Sparsity)

- *Let all of our assumptions be satisfied and let $\bar{u}$ be a locally optimal control. Then a Lagrange multiplier $\bar{\mu} \in L^\infty(\Sigma)$ and some $\kappa_0 > 0$ exist such that a.e. in Σ*

$$\bar{u}(x,t) = 0 \;\Leftrightarrow\; |\bar{\psi}(x,t)| \le \kappa \quad \forall \kappa \ge \kappa_0.$$

- *There is $\kappa_1 > 0$, such that $\bar{u} = 0 \quad \forall \kappa \ge \kappa_1$.* $\leftarrow \|\bar{\mu}_\kappa\|_\infty \le K \; \forall \kappa$

- *The element $\bar{\lambda}$ of the subdifferential $\partial j(\bar{u})$ is given by*

$$\bar{\lambda}(x,t) = \mathbb{P}_{[-1,1]}\left\{-\frac{1}{\kappa}\bar{\psi}(x,t)\right\}.$$

Outline

Duality theory and existence of $\bar{\mu}$

$$-a(x,t) := \bar{\varphi}(x,t) + \nu\bar{u}(x,t) + \bar{\lambda}(x,t)$$

$\bar{v} := \bar{u} - u_a$ solves the *primal problem*

$$\max \int_\Sigma a(x,t)v(x,t)\,d\sigma dt$$

$$v \le b + Sv$$

$$v \ge 0.$$

Have $b := u_d + \bar{y} + S(u_a - \bar{u}) - u_a \ge 0$!

$\bar{\mu}$ solves the *dual problem*

$$\min \int_Q b(x,t)\,\mu(x,t)\,d\sigma dt$$

$$\mu \ge a + S^*\mu$$

$$\mu \ge 0.$$

(provided that a solution exists)

Main steps:

- Equality of primal and dual optimal value ← boundedness condition, separation theorem
- Existence of an optimal $\bar{\mu}$ of the dual problem ← monotone behaviour, $b \ge 0$
- Uniform boundedness of μ w.r.t. to κ ← a bit technical

Existence of $\bar{\mu}$

Dual problem:

$$\min \int_Q b(x,t)\,\mu(x,t)\,d\sigma dt$$

$$\mu \geq a + S^*\mu \qquad \text{a.e. in } \Sigma$$

$$\mu \geq 0. \qquad \text{a.e. in } \Sigma.$$

Since $b \geq 0$ and $S^* \geq 0$, we find the solution $\bar{\mu}$ from

$$\mu = (a + S^*\mu)_+.$$

Dual linear programming problems in Hilbert space

Given:

- Real Hilbert spaces $\{U, \|\cdot\|_U\}$, $\{V, \|\cdot\|_V\}$,
- $A : U \to V$ linear and continuous,
- $K_U \subset U$ and $K_V \subset V$ nonempty, convex and closed "nonnegative" cones,
- partial orderings $u \geq_U 0$ iff $u \in K_U$, analogously $v \geq_V 0$ iff $v \in K_V$,
- elements $a \in U$ and $c \in V$.

Primal problem

$$\max\,(a\,,\,u)_U \quad \text{subject to} \quad Au \leq_V c, \quad u \geq_U 0. \qquad (\mathcal{PP})$$

Dual problem

$$\min\,(c\,,\,v)_V \quad \text{subject to} \quad A^*v \geq_U^* a, \quad v \geq_V^* 0. \qquad (\mathcal{DP})$$

Boundedness condition

Define for (variable) $d \in V$ and $e \in U$ the set

$$P(d) \quad = \quad \{u \in U : u \geq_U 0, \quad Au \leq_V d\},$$

Boundedness condition

$\exists \eta > 0$ independent of d and, for all $d \in V$, a closed set $K(d) \subset P(d)$ such that:

$$\|u\|_U \leq \eta \, \|d\|_V \quad \forall u \in K(d)$$

$$\forall u \in P(d) \; \exists \hat{u} \in K(d) : \; (a\,,\, u)_U \leq (a\,,\, \hat{u})_U.$$

Duality theorem

In our case: $U = V = L^2(\Omega), \qquad A = I - S.$

$$P(d) = \{u \in L^2(\Omega) : u \geq 0, \quad u - Su \leq d\}$$

The boundedness condition is easily verified.

Theorem (Strong duality)

If the feasible set $P(c)$ of the primal problem $(\mathcal{PP})$ is nonempty and the boundedness condition is satisfied, then the primal problem has an optimal solution. Moreover, the strong duality relation

$$\max_{u \in P(c)} (a\,,\,u)_U = \inf_{v \in D(a)} (c\,,\,v)_V$$

is fulfilled.

If the boundedness condition is satisfied, then the following cone is closed:

$$K = \{(d, \delta) \in V \times \mathbb{R} : \exists u \in P(d) \text{ with } (a\,,\,u)_U \geq \delta\}.$$

References

E. Casas, F. T.
Sparse optimal control for the heat equation with mixed control-state constraints
2018, To appear in Mathematical Control and Related Fields

E. Casas, F. T.
Optimal sparse boundary control for a semilinear parabolic equation with mixed control-state constraints
2019, submitted

F. T.
Existenz- und Dualitätsaussagen für lineare Optimierungsaufgaben in reflexiven Banachräumen
Math. Operationsforschung und Statistik 6, 1975

Best wishes, Jean-Pierre!

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